Cut-off properties in asynchronous probabilistic distributed protocol

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Sixth Cassting Meeting 2015

Definition (Distributed protocol)

A distributed protocol is given by $\mathcal{P} = \langle Q, D, (q_0, d_0), T, q_f \rangle$

- Q: control states
- D: possible values of the register
- $(q_0, d_0) \in Q \times D$: initial state with an initial value
- T: transitions of the form $p \xrightarrow{r(d)} q$ and $p \xrightarrow{w(d)} q$ for $p, q \in Q$, $d \in D$.
- q_f: final state

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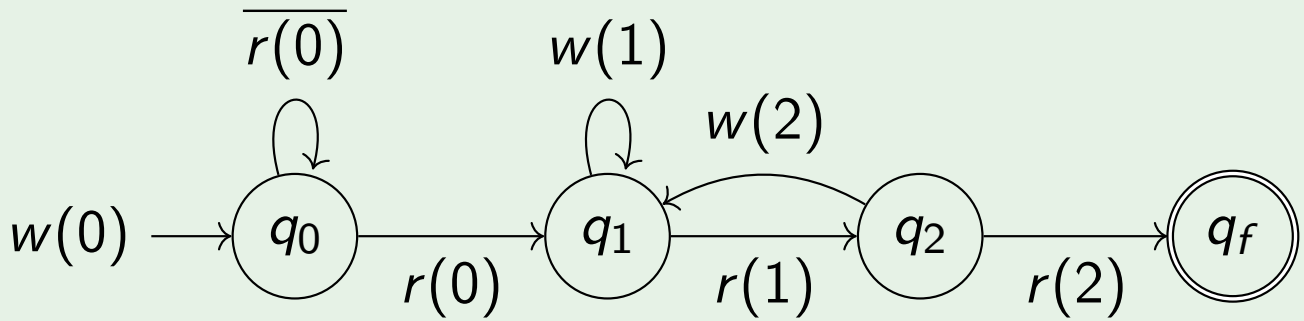

Definition (Configuration of the protocol)

$$\gamma = (f, d)$$

with $f : Q \to \mathbb{N}$ (multiset) and $d = v(\gamma) \in D$ the register value.

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Definition (Semantics)

$(f, d) \to (f', d')$ if $f' = f - q + q'$ with either

- $q \xrightarrow{w(d')} q'$
- or $d = d'$ and $q \xrightarrow{r(d)} q'$

Markov Chain

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Qualitative goal

Estimate $\mathbb{P}_n(\diamond q_f)$ $(\diamond q_f = \{\gamma \mid \gamma(q_f) > 0\})$

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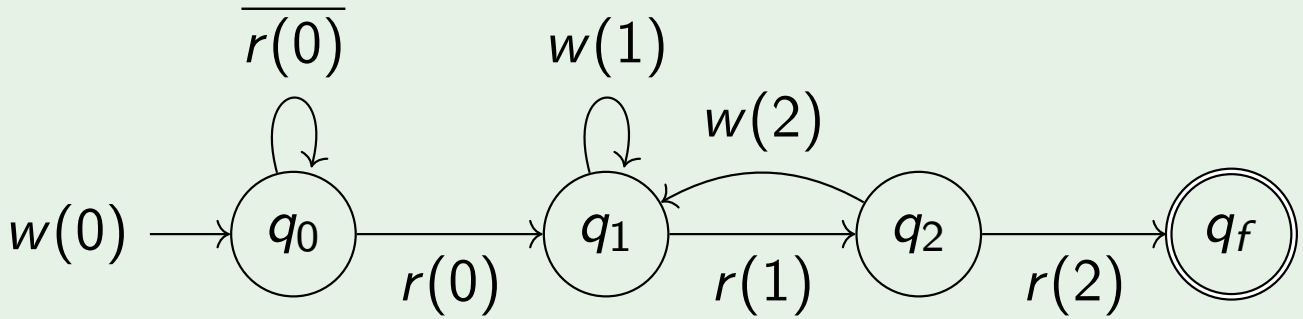

$$\longrightarrow 0, q_0^3 \longrightarrow 0, q_0^2 q_1 \to 0, q_0^1 q_1^2 \longrightarrow 0, q_1^3 \longrightarrow 1, q_1^3 \longrightarrow 1, q_1^2 q_2 \to 1, q_1 q_2^2$$

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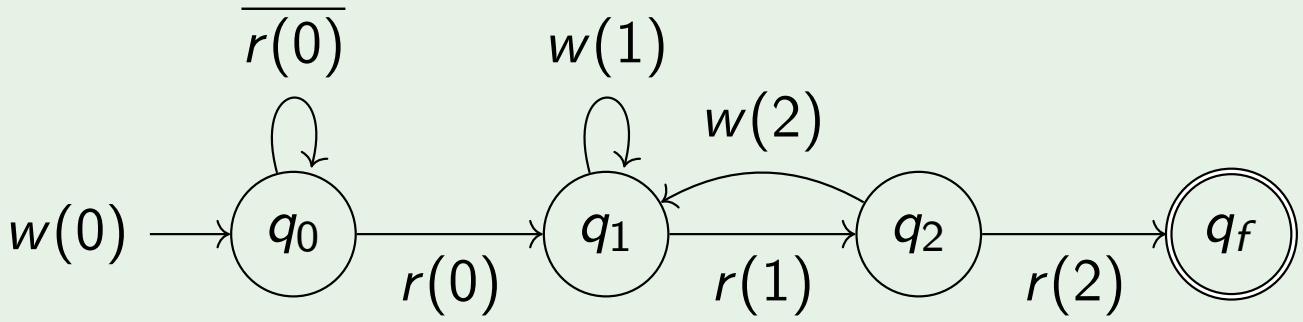

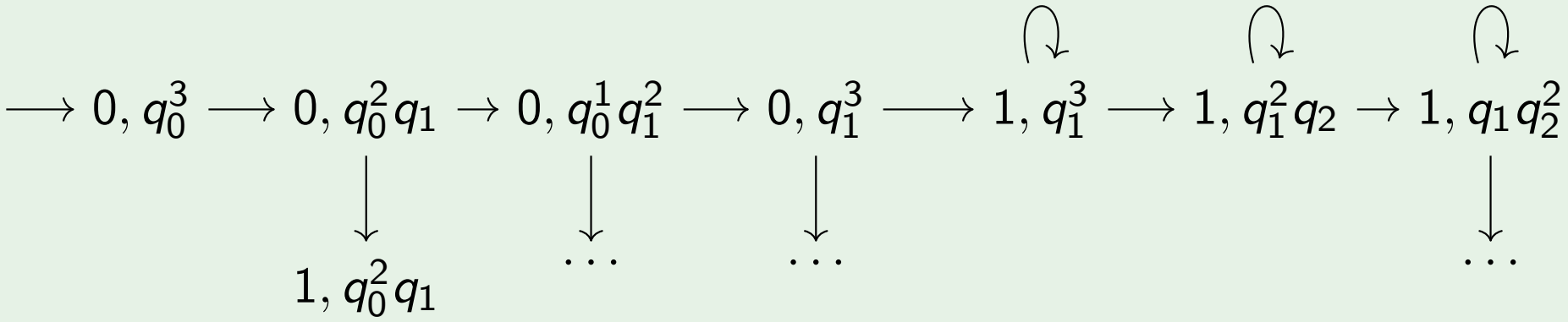

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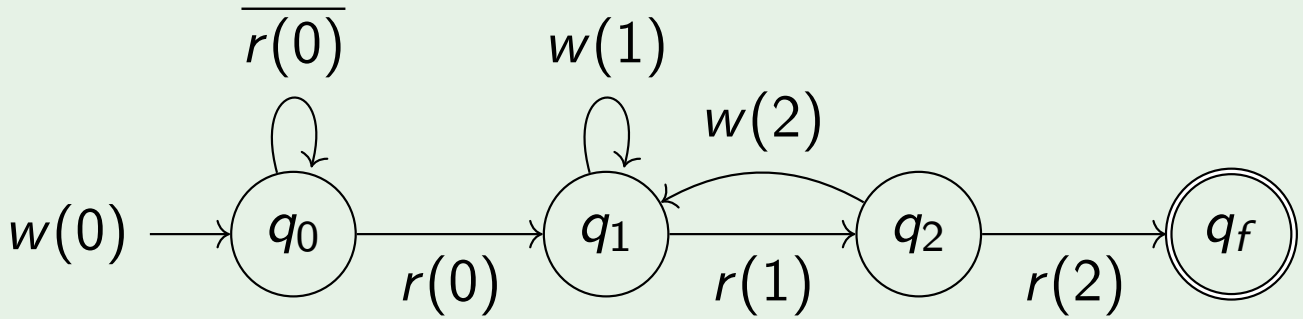

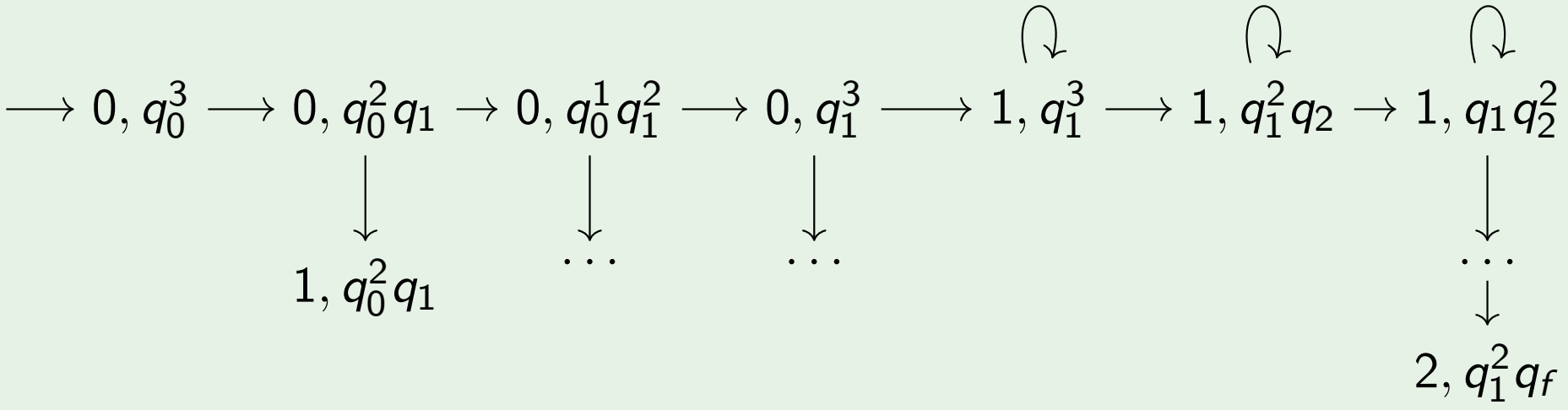

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For $B \in \{\{0\}, (0,1), \{1\}\}$, the property $\mathbb{P}_n(\diamond q_f) \in B$ does not depend on the actual distributions

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- Both the scheduler and processes are ~~non-deterministic~~ stochastic
- No atomicity
- For fixed parameter n, can be encoded as a Petri Net

What we are looking for

Definition

Let $B \in \{\{0\}, (0,1), \{1\}\}$ and N such that $\forall n \geq N \ \mathbb{P}^n(\diamond q_f) \in B$. Then N is a cut-off.

Symbolic graph

Definition (Symbolic graph)

$G_{\text{symb}} = (S, E)$ with

- $S = 2^Q \times D$
- E is defined by $(X_1, d_1) \to (X_2, d_2)$ if there exists $x_1 \in X_1$, $x_2 \in X_2$ such that

$$X_1 \backslash \{x_1, x_2\} = X_2 \backslash \{x_1, x_2\} \tag{1}$$

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Lemma

Every "concrete" run of $\mathcal{P}$ corresponds to a symbolic run.

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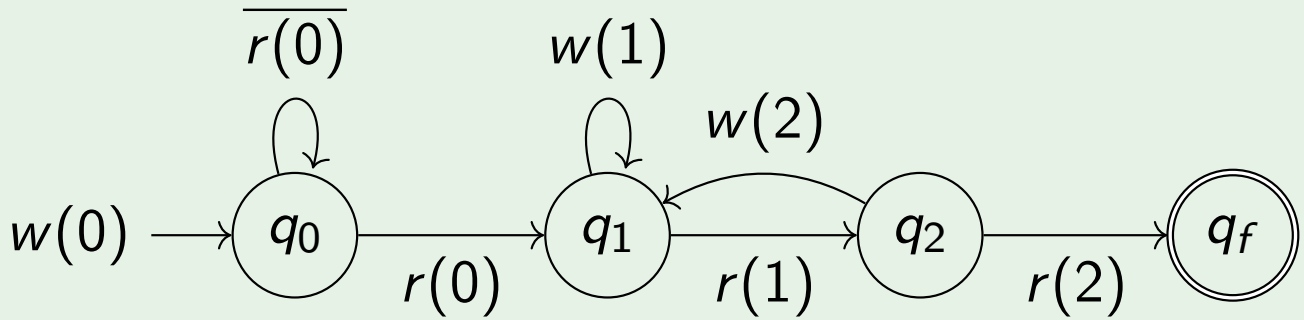

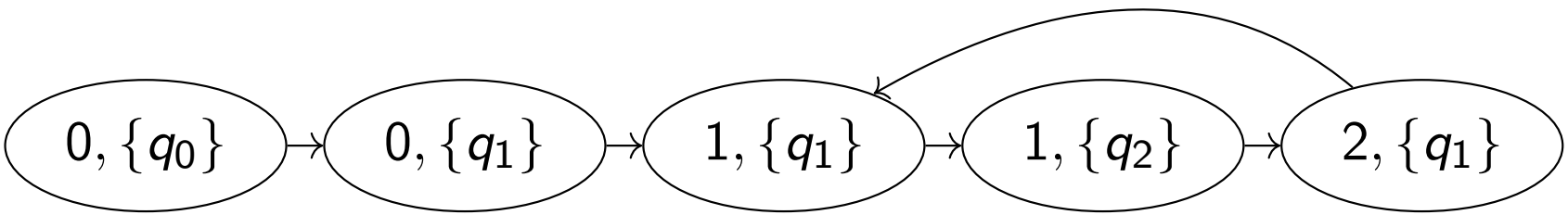

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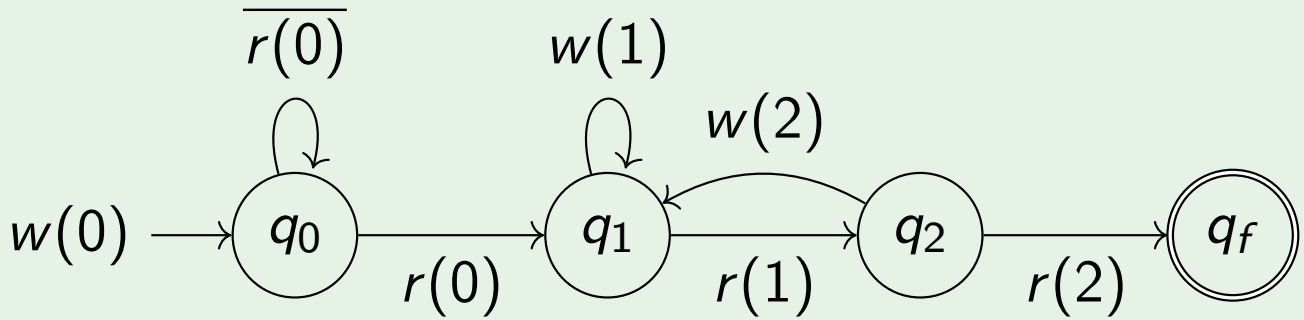
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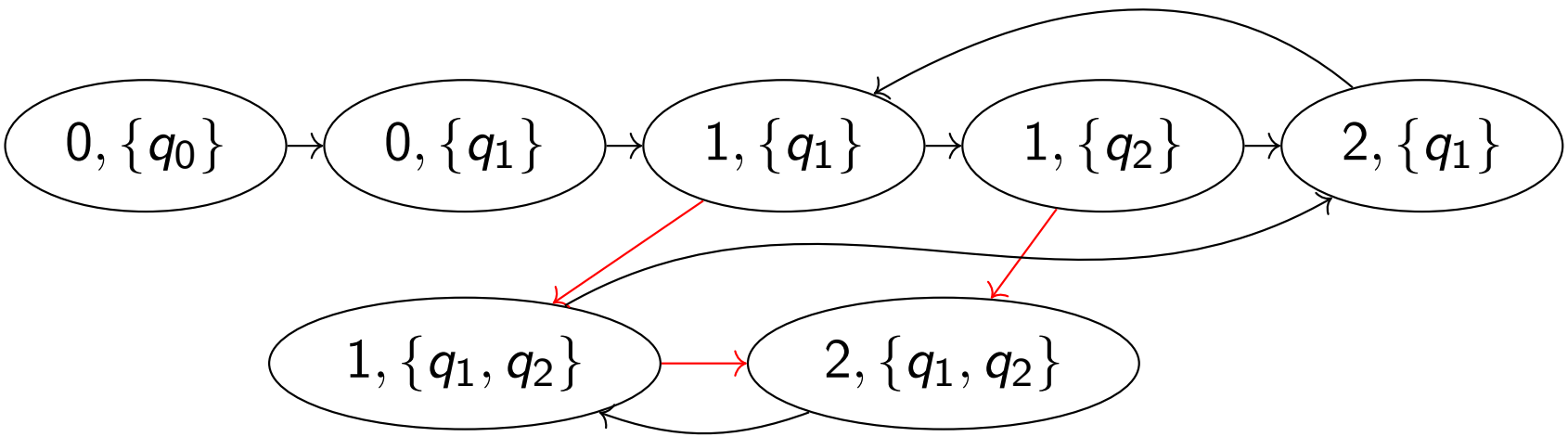
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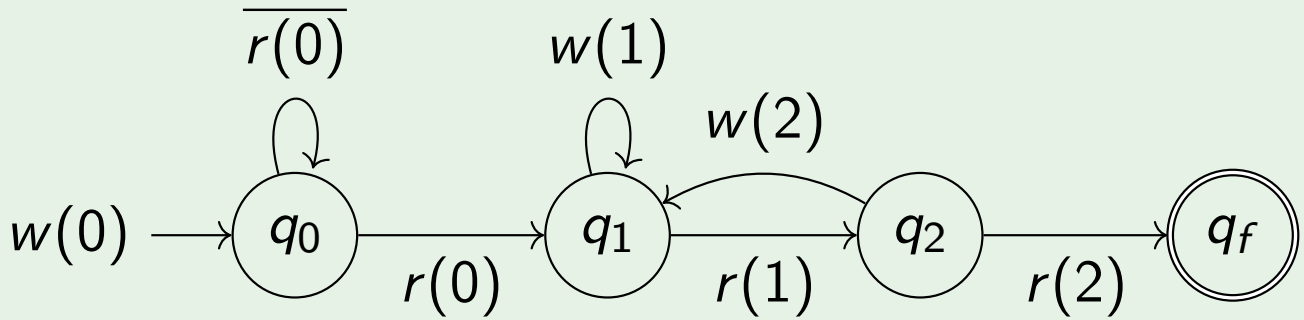
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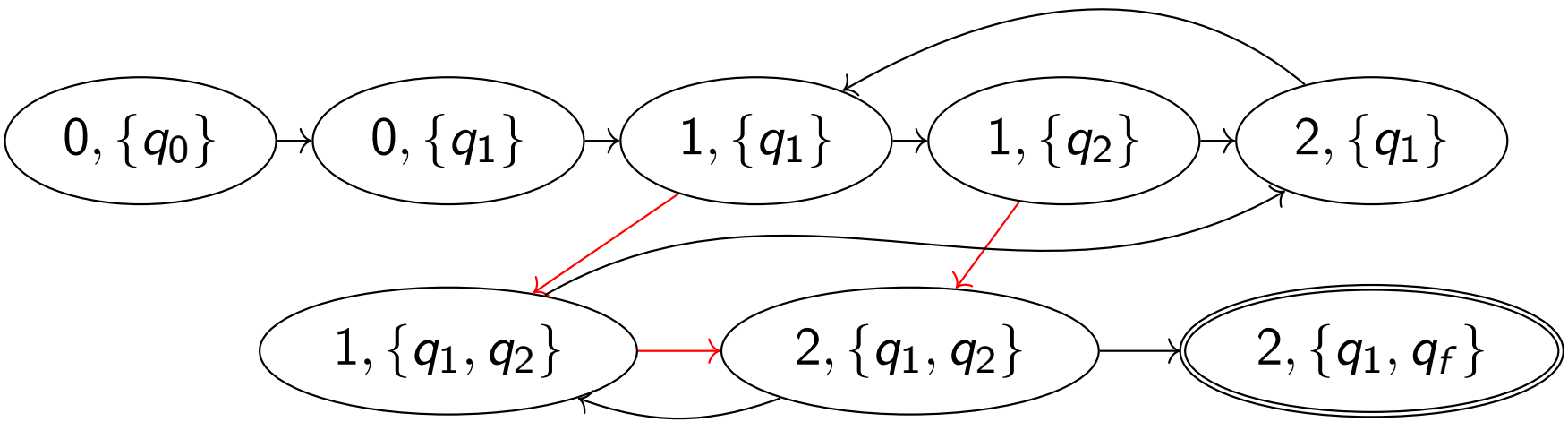
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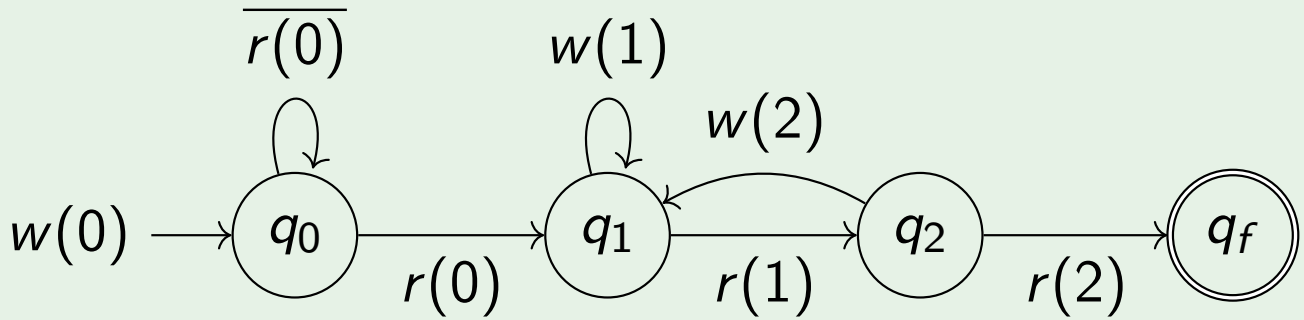
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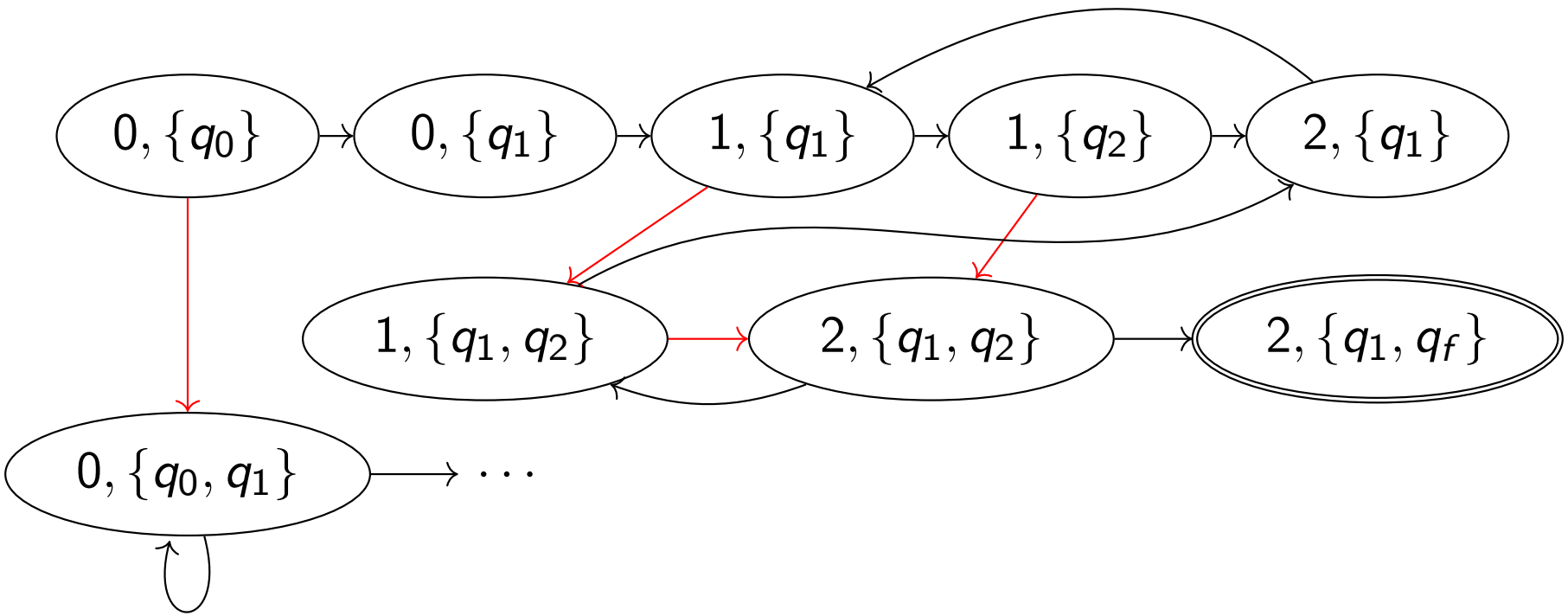
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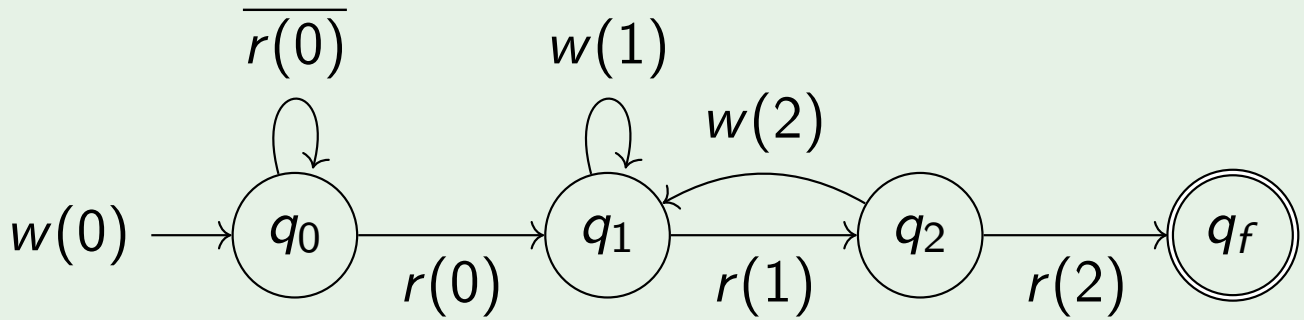
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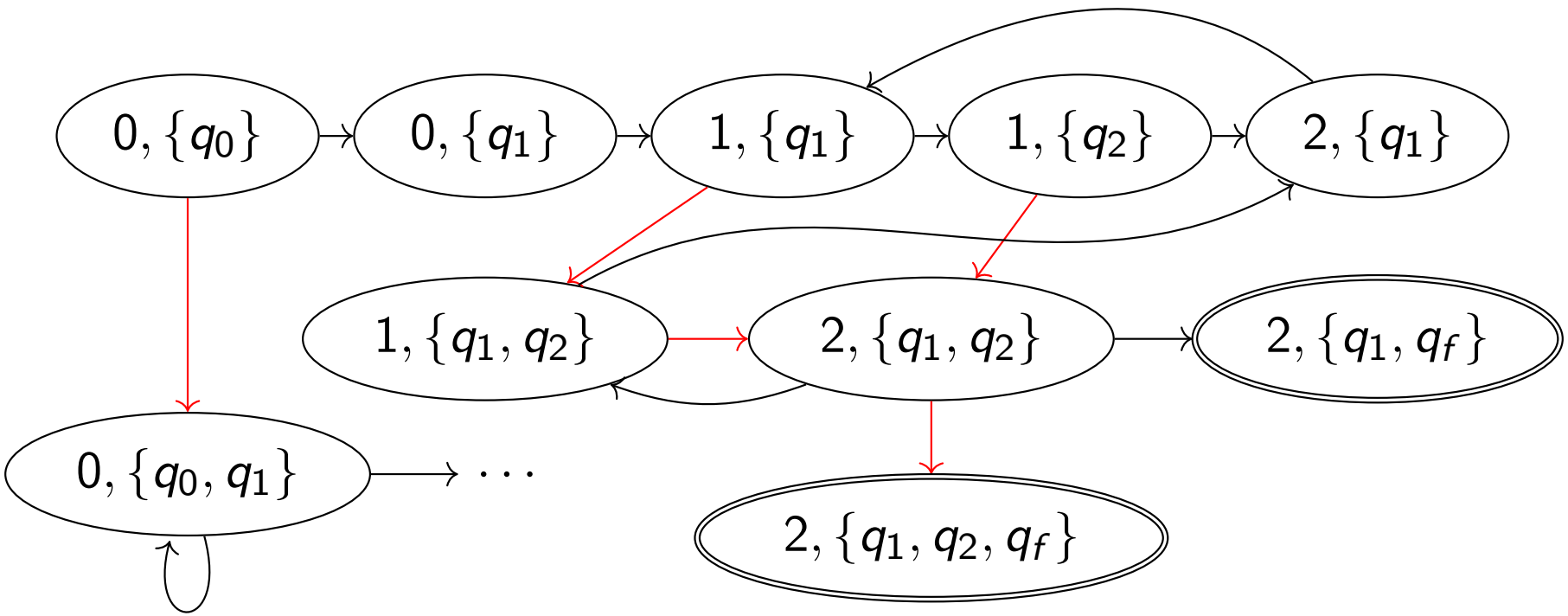
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Reconstructing the run

Lemma

If $(X, d) \rightarrow^{L} (Y, e)$ *in* G_{symb}, *then there exists a concrete run* $\gamma \rightarrow^{*} \gamma'$ *with*

- $S(\gamma) = X$ *and* $v(\gamma) = d$
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Sketch.

Every red transition in the symbolic graph implies a copy of the current involved state. □

Negative cut-off: the easy case

Remark

If $\mathrm{Succ}^*((\{q_0\}, d_0)) \not\subseteq \mathrm{Pre}^*(\diamond q_f)$ in G_{symb}, then $\mathbb{P}^n(\diamond q_f) < 1$ for n large enough (negative cut-off).

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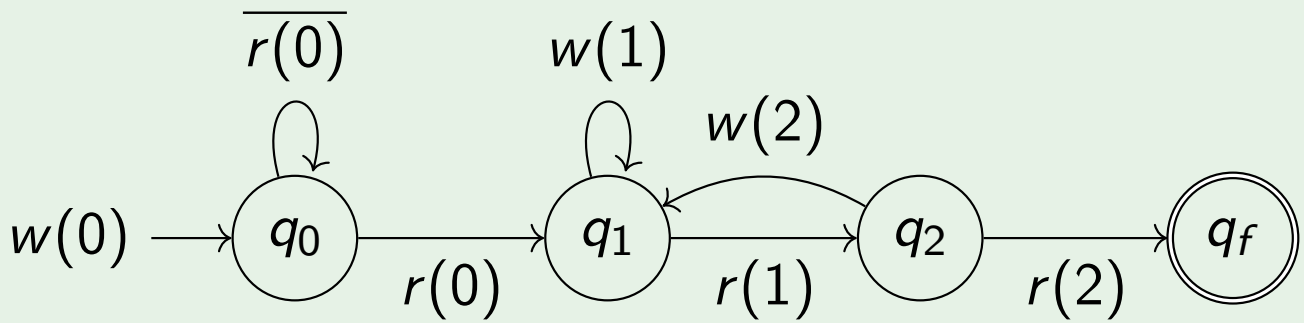

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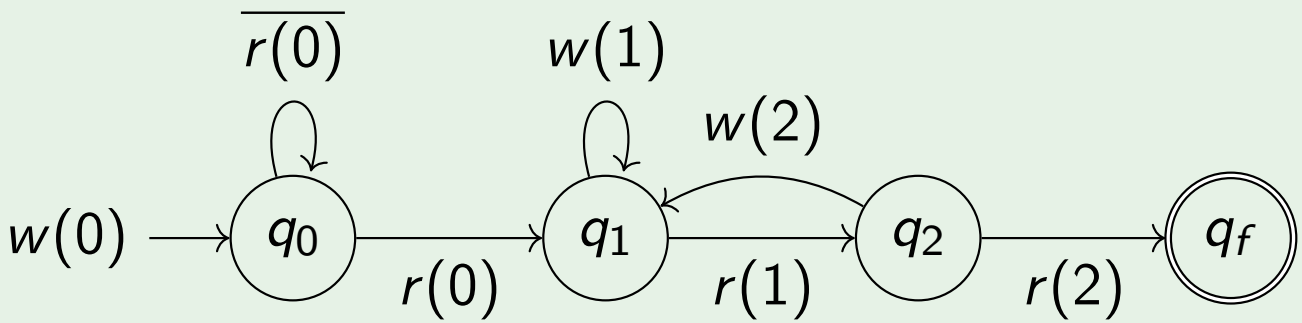
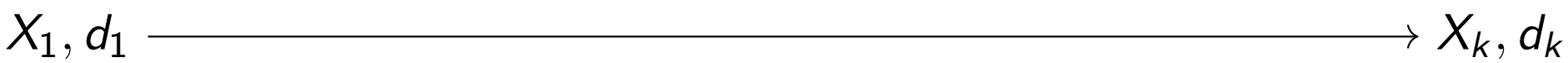


$$(q_0^n, 0) \xrightarrow{r(0)} (q_0^{n-1} q_1, 0) \xrightarrow{w(0)} (q_0^{n-1} q_1, 1) \not\to^* \diamond q_f$$

Reducing a symbolic path



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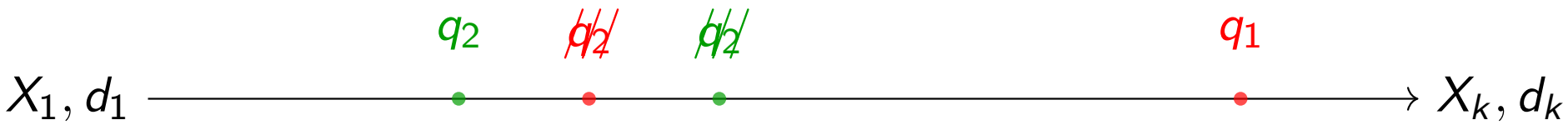
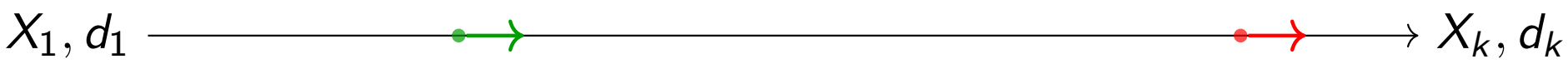


Reducing a symbolic path

Without loss of generality

For each $q \in Q$, there is at most two transitions making q appearing or disappearing (one each).

Fixed support behaviour



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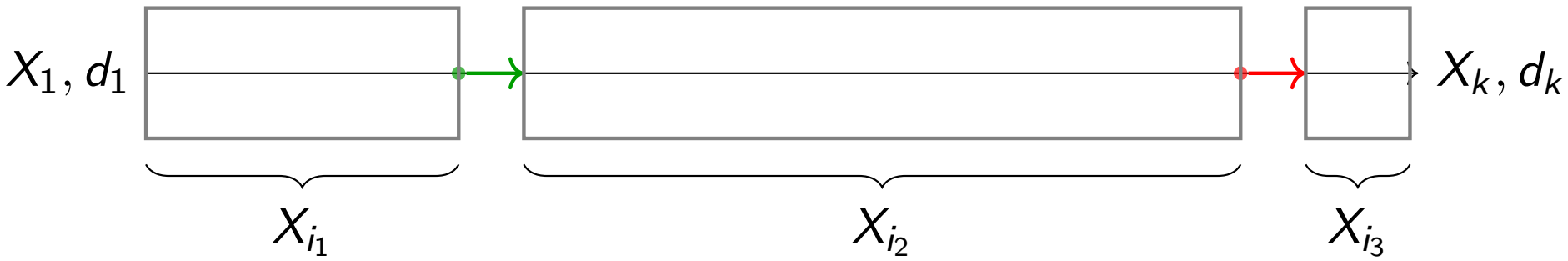

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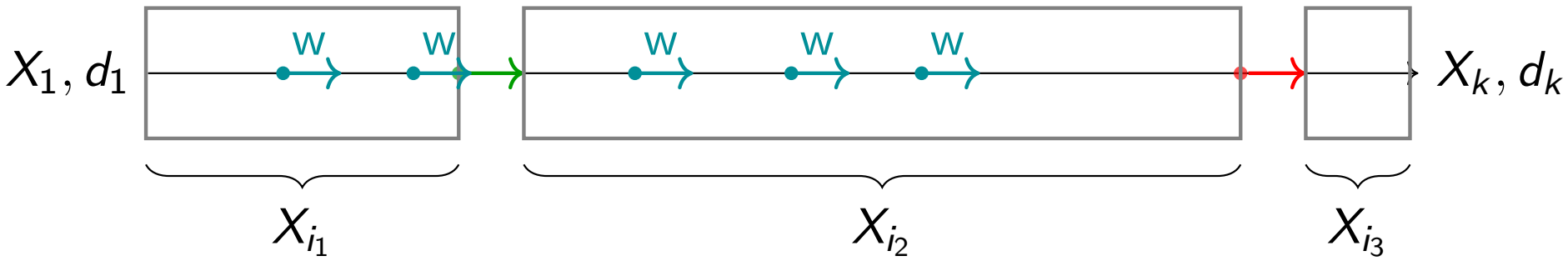

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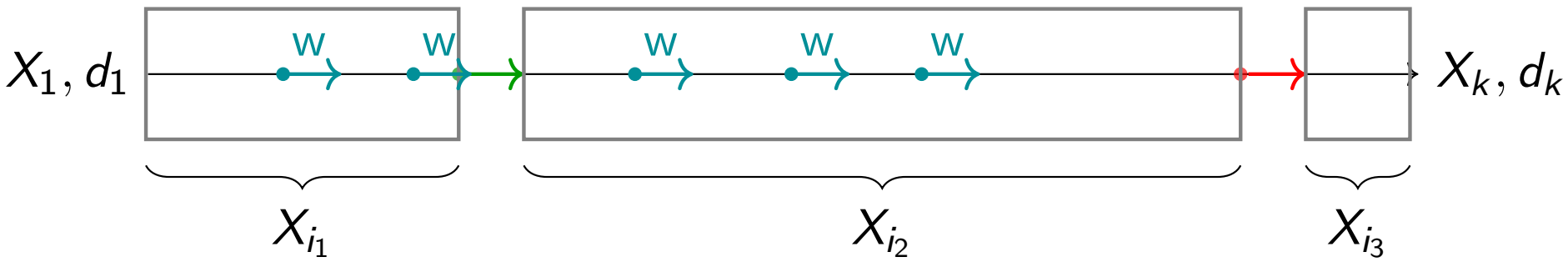

Key Idea

Only the last write transition is required.

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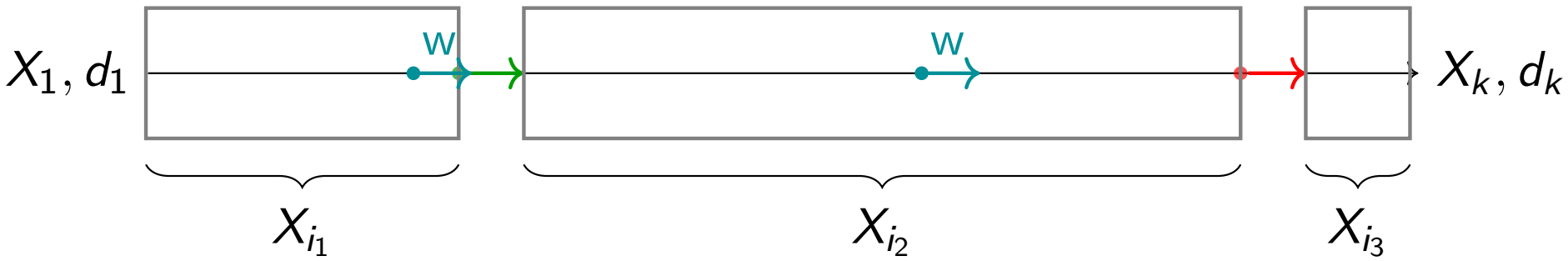

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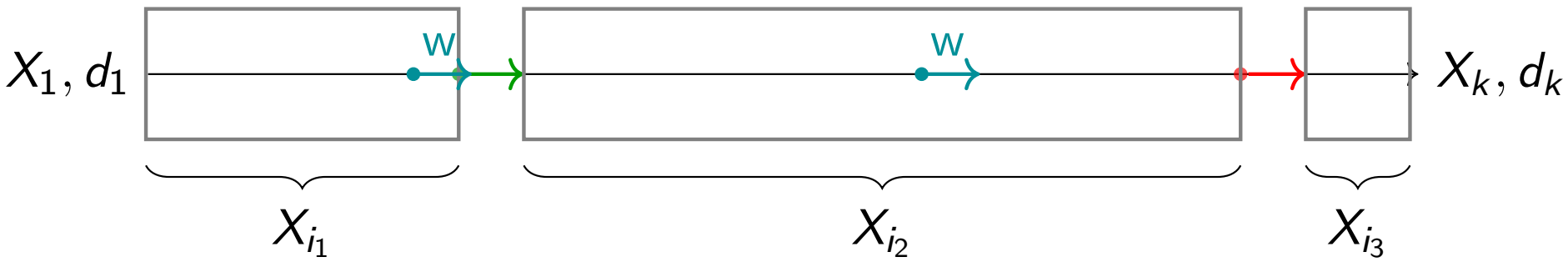

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Theorem

Every path in G_{symb} *has less than* $4|Q| + 1$ *transitions.*

What the symbolic graph taught us

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This solves the cut-off existence problem in the case $\{0\}$ since

$$\mathbb{P}^n(\diamond q_f) > 0 \;\Rightarrow\; \mathbb{P}^{\min(n,4|Q|+1)}(\diamond q_f) > 0$$

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It remains to determine the 2 others cut-offs:

- The "positive" case : $\mathbb{P}^{\geq N}(\diamond q_f) \equiv 1$
- The "negative" case : $\mathbb{P}^{\geq N}(\diamond q_f) < 1$

Upward closed sets and well-quasi-orders

Definition (Well-quasi-ordering (wqo))

A quasi-ordering $\preceq$ on a set X such that any infinite sequence of elements $x_0, x_1 \ldots$ from X contains an increasing pair $x_i \preceq x_j$ with $i < j$.

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Main idea: express $\mathrm{Pre}^*(\diamond q_f)$ and $\mathrm{Succ}^*((q_0^k, d_0))$ as upward-sets to "discretize" the problem.

The right partial order

Definition

$$\gamma \preceq \gamma' \Leftrightarrow v(\gamma) = v(\gamma') \wedge \forall q\ \gamma(q) \leq \gamma'(q)$$

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We write

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What about the negative cut-off case ?

Negative case

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How does

$$\min \mathrm{Succ}^*((q_0^{\geq n}, d_0)) = \{\gamma_{1,n} \dots \gamma_{k_n,n}\}$$

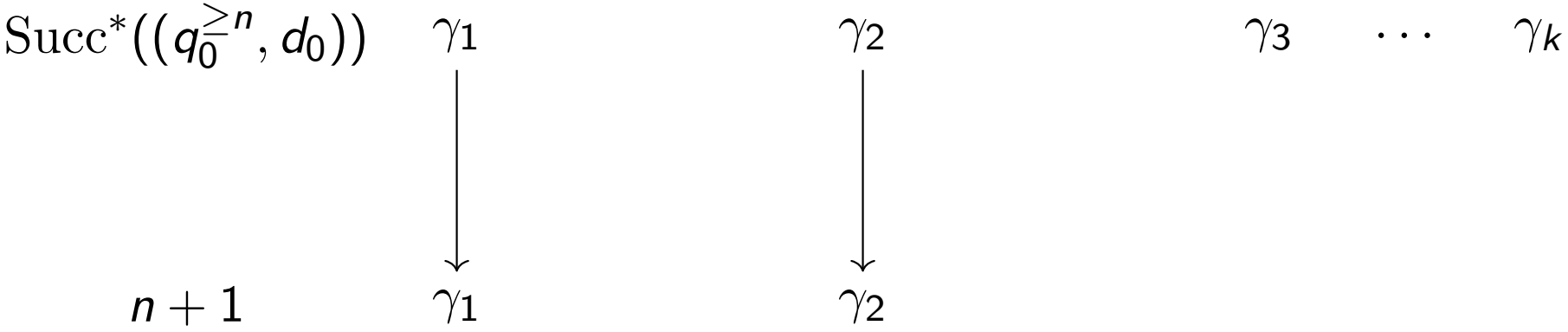
varies w.r.t n ?

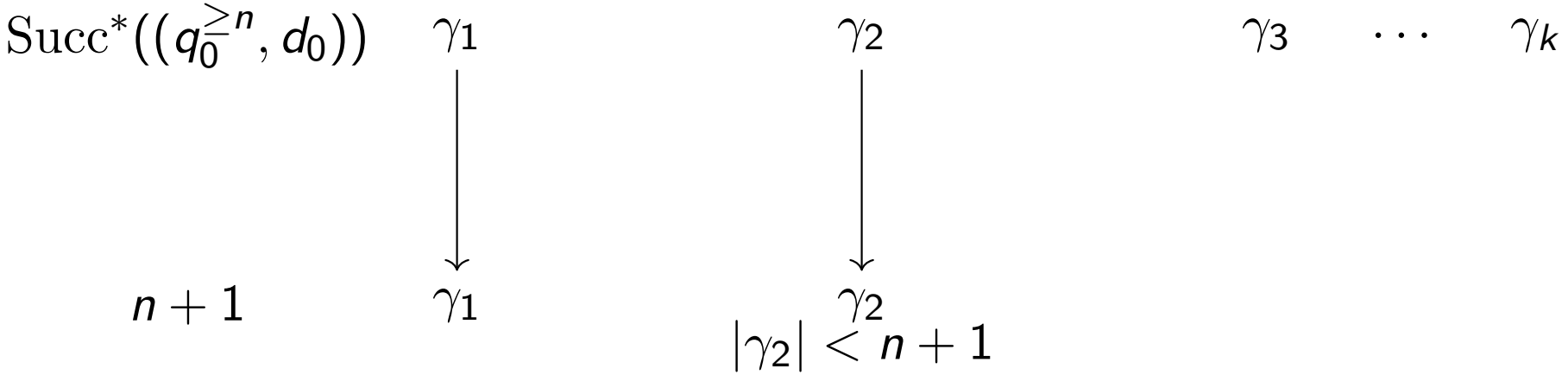
$\mathrm{Succ}^*((q_0^{\geq n}, d_0))$ $\quad \gamma_1 \quad \gamma_2 \quad \gamma_3 \quad \cdots \quad \gamma_k$

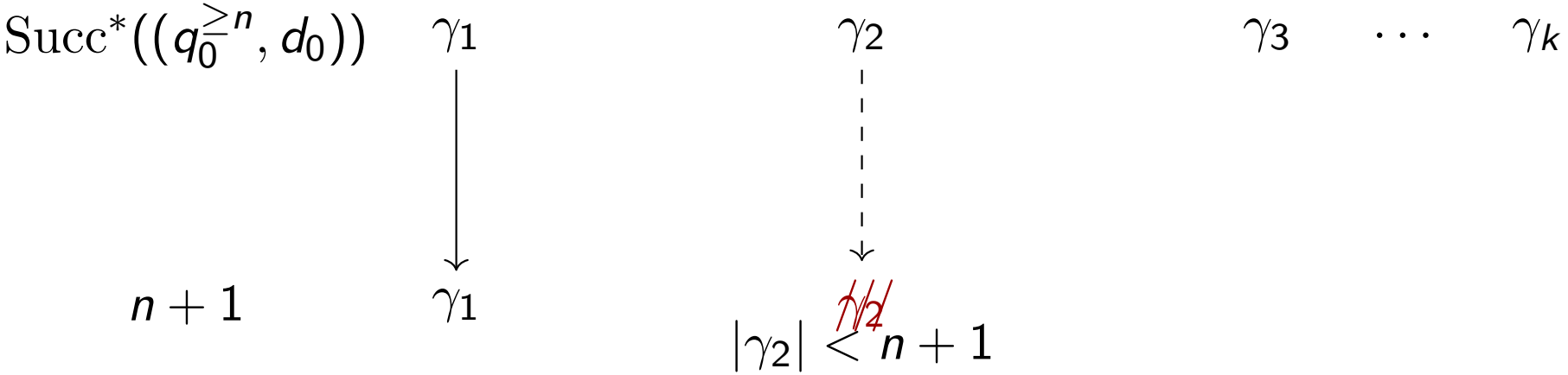
$\mathrm{Succ}^*((q_0^{\geq n}, d_0))$ γ_1 γ_2 γ_3 $\cdots$ γ_k

$n+1$

$$\begin{array}{lllll} \mathrm{Succ}^*((q_0^{\geq n}, d_0)) & \gamma_1 & \gamma_2 & \gamma_3 \quad \cdots & \gamma_k \\ & \downarrow & & & \\ n+1 & \gamma_1 & & & \end{array}$$


$\mathrm{Succ}^*((q_0^{\geq n}, d_0))$
γ_1
γ_2
γ_3
$\cdots$
γ_k
$n+1$
γ_1
γ_2


$\mathrm{Succ}^*((q_0^{\geq n}, d_0))$
γ_1
γ_2
γ_3
$\cdots$
γ_k
$n+1$
γ_1
γ_2
$|\gamma_2| < n+1$


$\mathrm{Succ}^*((q_0^{\geq n}, d_0))$
γ_1
γ_2
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$n+1$
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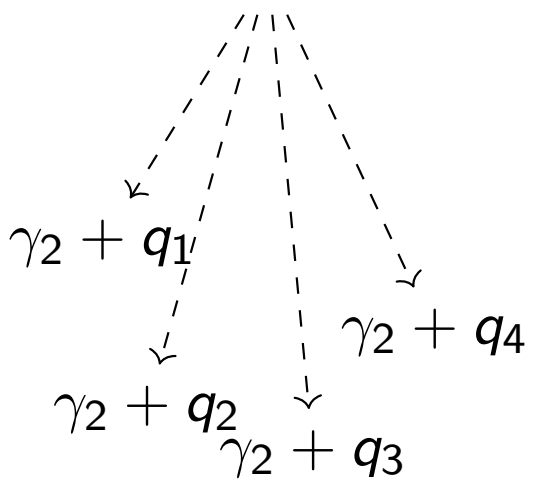
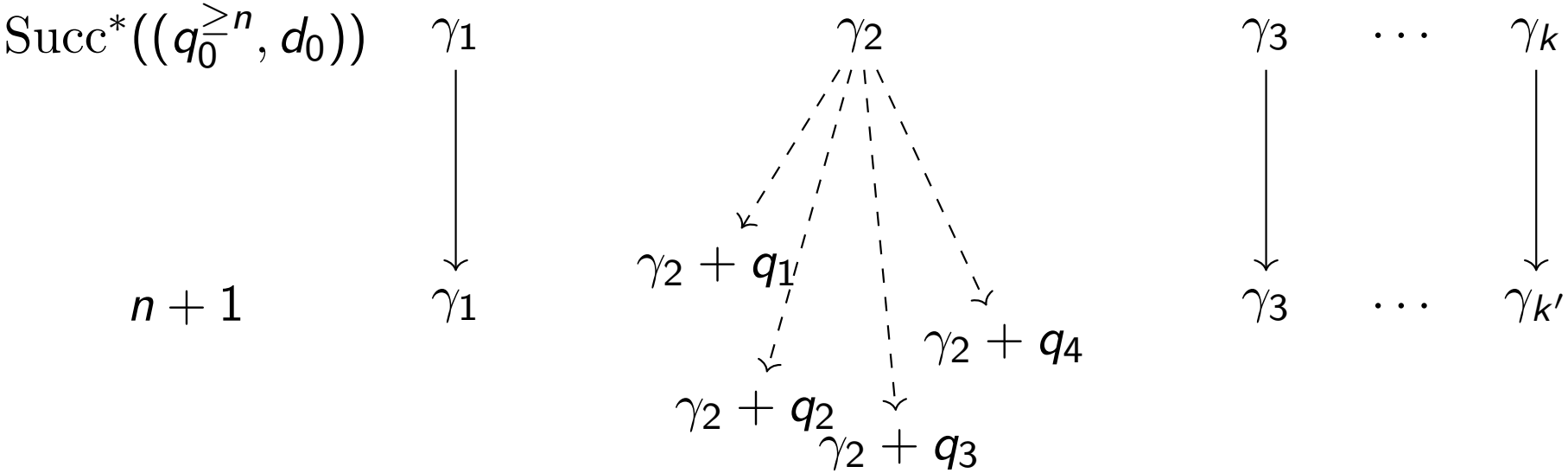
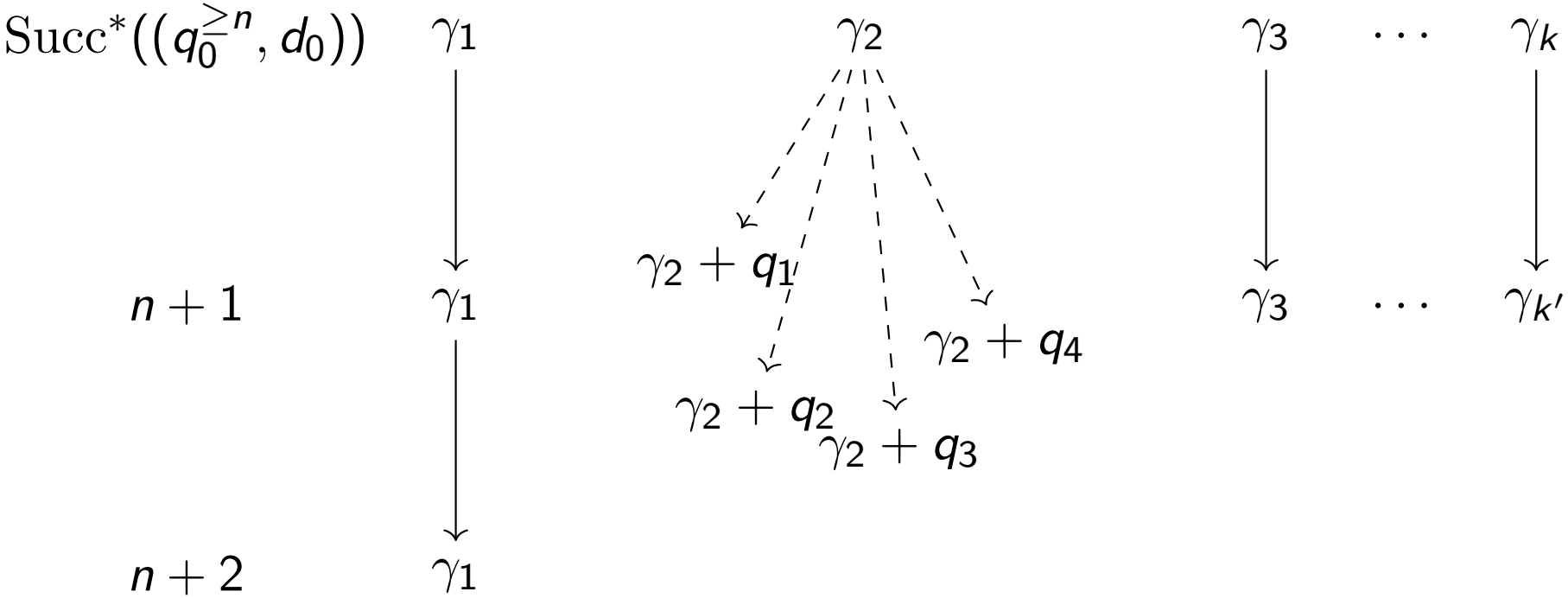


$n+1$ γ_1

$q_i \in S(\gamma_2)$


$\mathrm{Succ}^*((q_0^{\geq n}, d_0))$
γ_1
γ_2
γ_3
$\cdots$
γ_k
$\gamma_2 + q_1$
$\gamma_2 + q_4$
$\gamma_2 + q_2$
$\gamma_2 + q_3$
$n+1$
γ_1
γ_3
$\cdots$
$\gamma_{k'}$


$\mathrm{Succ}^*((q_0^{\geq n}, d_0))$
γ_1
$n+1$
γ_1
$n+2$
γ_1
γ_2
$\gamma_2 + q_1$
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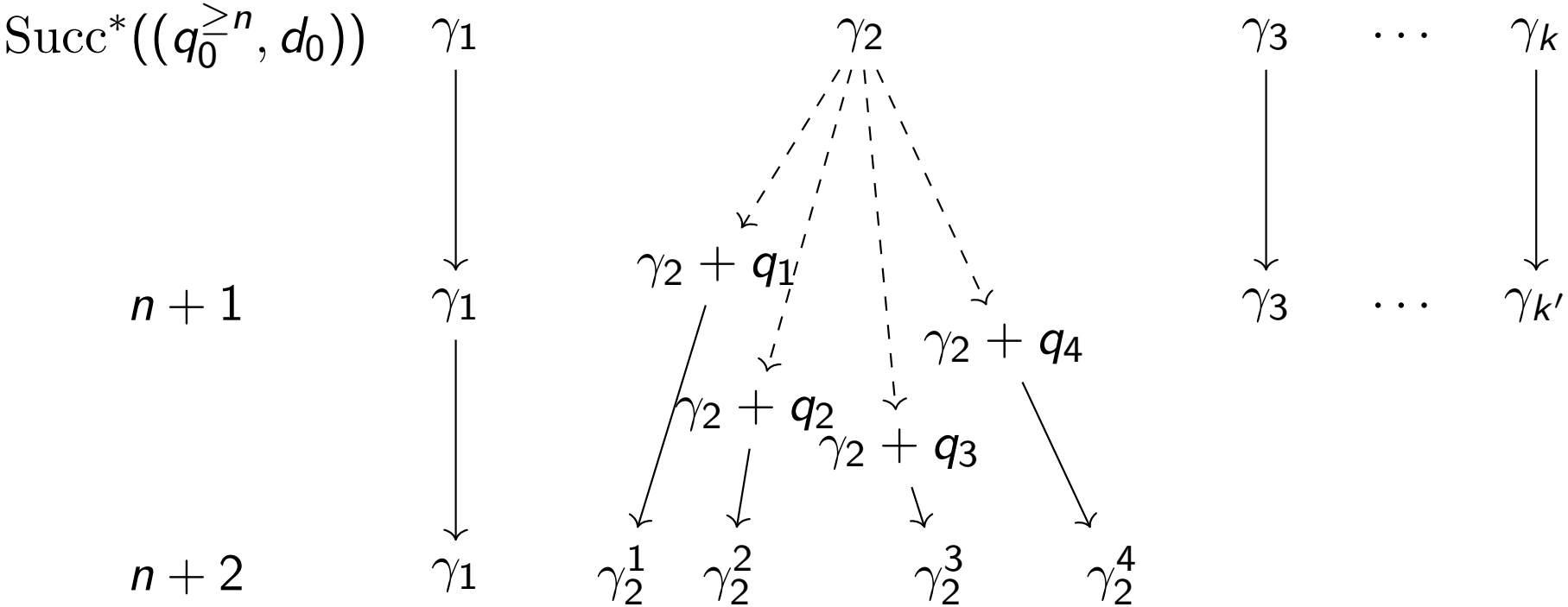
$\gamma_2 + q_4$
γ_3
$\cdots$
γ_k
γ_3
$\cdots$
$\gamma_{k'}$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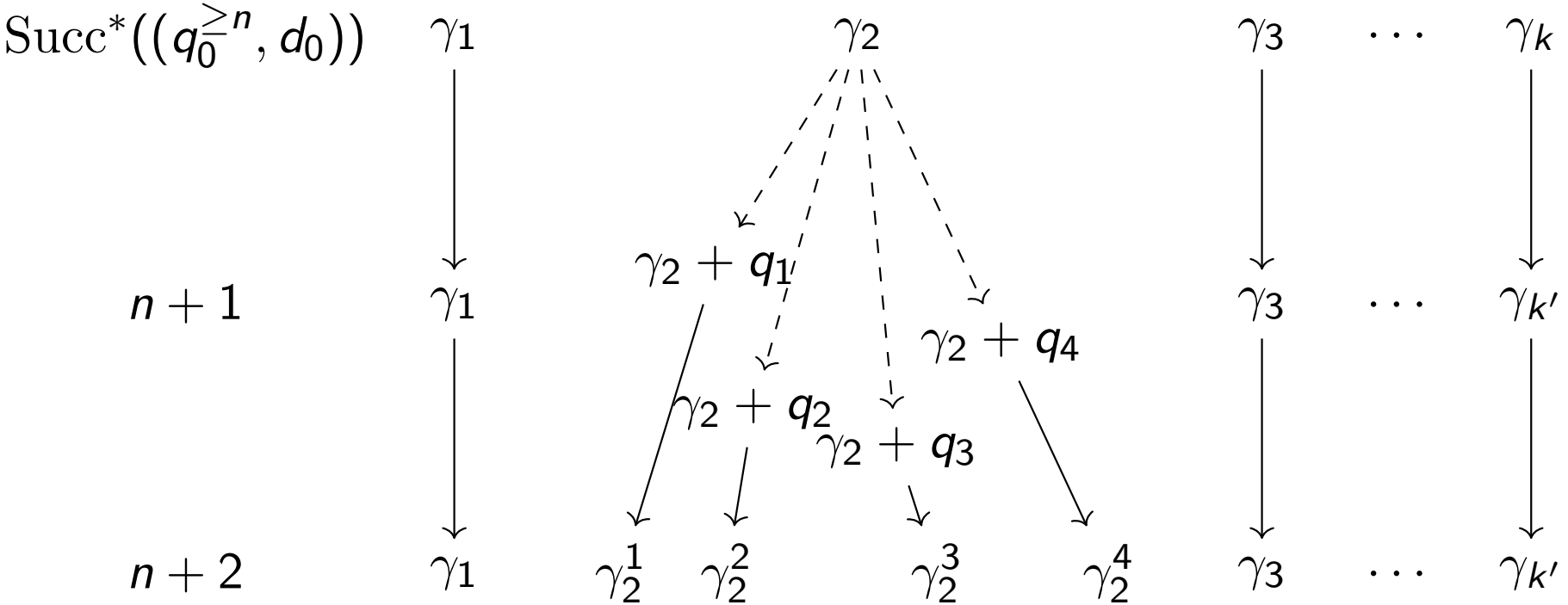
γ_3
$\cdots$
$\gamma_{k'}$
$n+2$
γ_1
γ_2^1
γ_2^2
γ_2^3
γ_2^4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Variation of the basis

1. Start with $X_1 = \{\gamma_1 \dots \gamma_k\} = \min \mathrm{Succ}^*((q_0^{\geq 1}, d_0))$

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Goal

Keep track of a "problematic" generator $\gamma \in X_k$ as $k \to \infty$.

Definition

Assume

$$\mathrm{Pre}^*(\diamond q_f) = \cup_{j=1}^{k} \uparrow \eta_j$$

and let $\gamma \in \mathrm{Succ}^*((q_0^{\geq 1}, d_0))$. *We define:*

$$A(\gamma) = \bigcap_{j=1}^{m} \left\{ q \in S(\gamma) \;\middle|\; \forall k \geq 0 \; \eta_j \not\preceq \gamma + q^k \right\}$$

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Theorem (Negative cut-off)

If $A(\gamma) \neq \emptyset$. *Then* $N = |\gamma|$ *is a negative cut-off.*

Positive cut-off

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Assume $A(\gamma) = \emptyset$. Then there exists $N(\gamma) \geq |\gamma|$ such that

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Proof.

For any $q \in S(\gamma)$, there exists j_q such that $q \notin \left\{ q \mid \forall k \geq 0 \ \eta_{j_q} \not\preceq \gamma + q^k \right\}$ so there exists k_q such that $\eta_{j_q} \preceq \gamma + q^{k_q}$.
Define $N(\gamma) = |\gamma| + \sum_{q \in S(\gamma)} k_q$. □

Existential solution

Theorem

Given a protocol $\mathcal{P}$ there always exists either a positive cut-off either a negative cut-off N.
The probability to reach $\diamond q_f$ is eventually 1 or eventually strictly less than 1.

- Non-constructive proof
- We never computed the γ_i and η_j
- If computed, we can give a polynomial bound of N in their size
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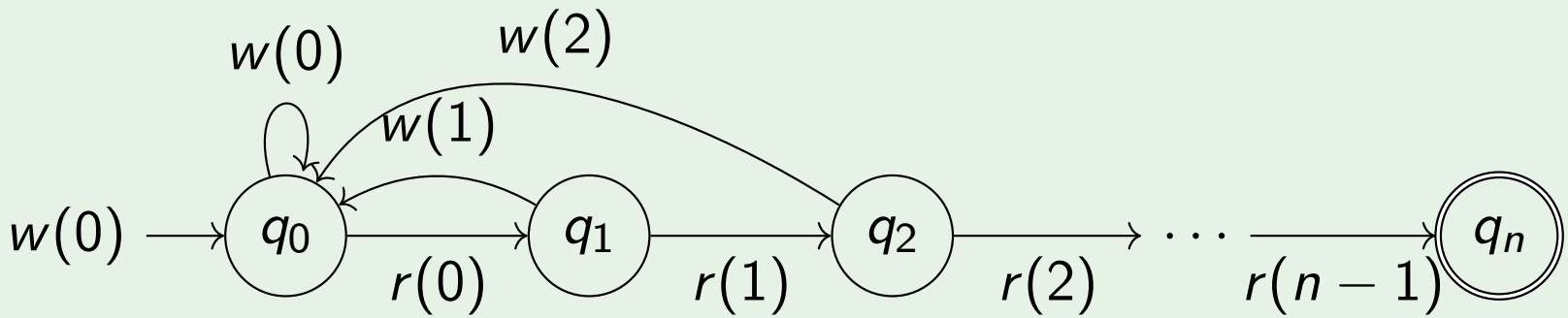
- Deciding the positive or negative case ?
- Simulate the Markov chain with N initial processes ?

Linear example

Example



Cut-off value ?

Linear example

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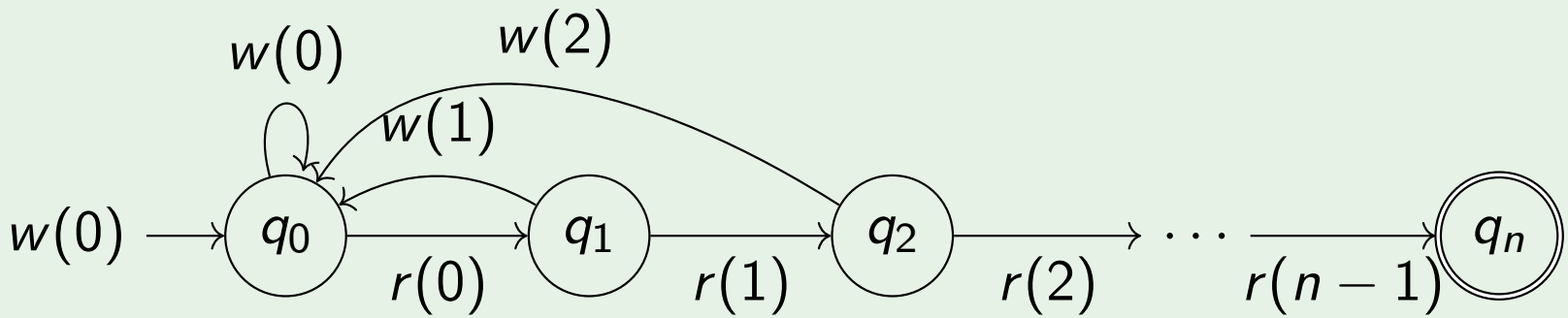

Cut-off value ? The cut-off is positive and equals n.

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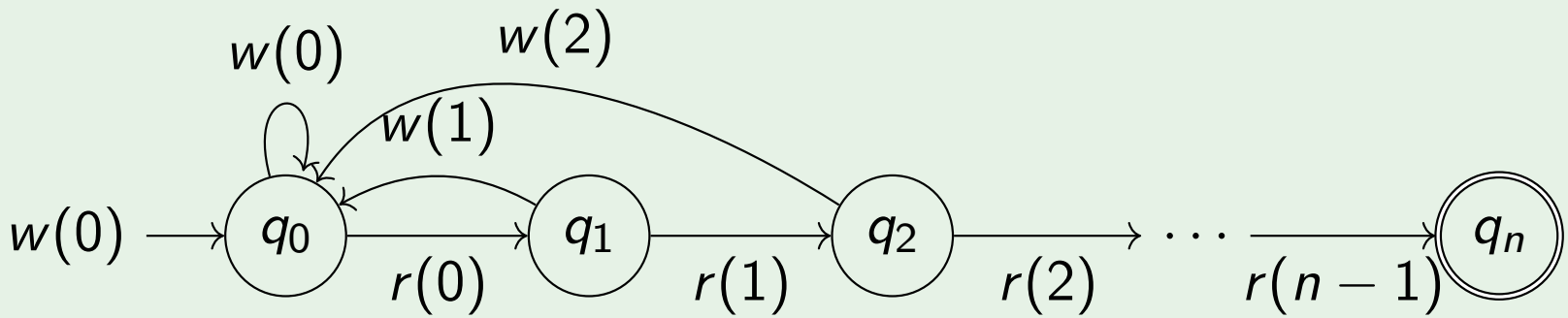

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Invariant:

$$\forall j \leq m \ \sum_{k=0}^{j} \gamma(q_k) \geq j + \mathbb{1}_{v(\gamma)=j+1}$$

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Decision Problem

- INPUT: a protocol $\mathcal{P}$
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Theorem

The cut-off decision problem is PSPACE-hard.

Sketch.

We reduce the halting of a linear bounded Turing machine $\mathcal{M}$.

- A given tape position i containing letter x is coded by a fixed state
- The current head position is coded in the register

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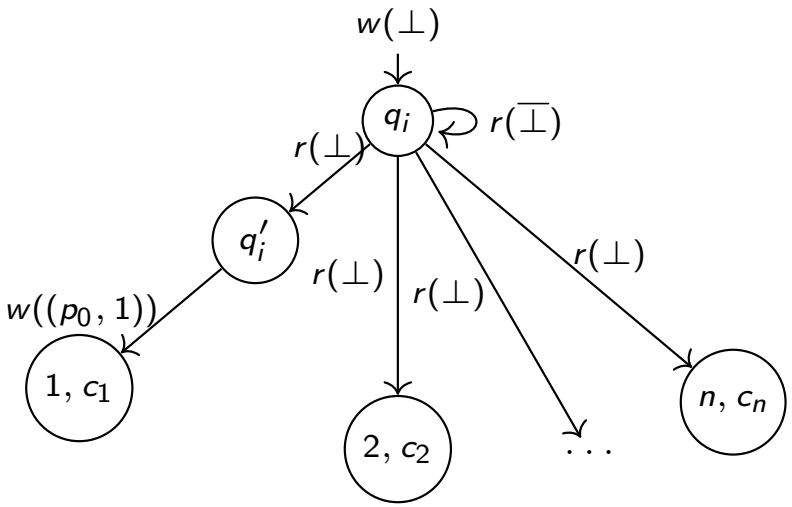
We reduce the halting of a linear bounded Turing machine $\mathcal{M}$.

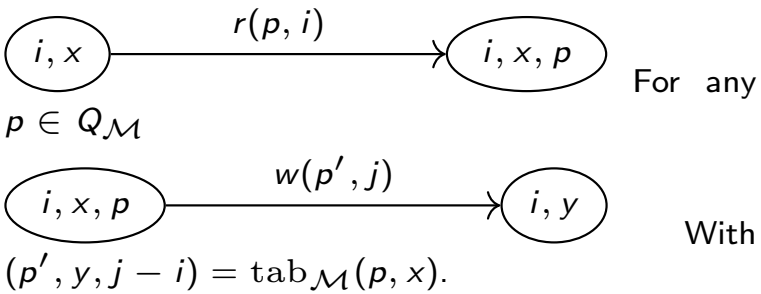
- A given tape position i containing letter x is coded by a fixed state
- The current head position is coded in the register
- If the number of states is too big, we cannot ensure proper encoding
- Key idea: we code improper encoding/non-termination by $\mathbb{P}(\diamond q_f) = 1$
- The previous module ensures $\diamond q_f$ if too many processes encoded the machine

We build $\mathcal{P}$ protocol such that

$$\mathbb{P}^{\geq n}(\diamond q_f) = 1 \iff \mathcal{M} \text{ does } \underline{\text{not}} \text{ terminate}$$

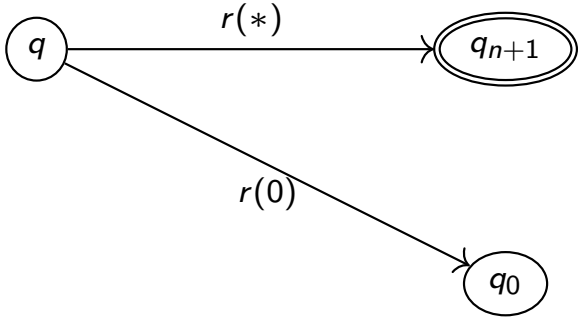


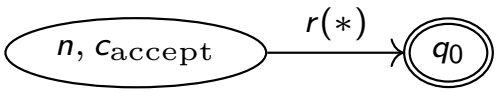
(a) $\{q_i\}$: Initialization: $c_1 \ldots c_n$ is the initial data band



(b) Q_S: Simulation step

For every $q \in Q_S$, we add the two following transitions:





(c) Q_S: Final transition (d) Q_S: two extra transitions that ensure leaving Q_S on the last memory cell almost surely

Figure: Reduction of a linear bounded Turing machine $\mathcal{M}$, the $q_0 \ldots q_{n+1}$ module is the "filter $n+1$ module"

Upper bound ?

- Current investigation: bound the cut-off by bounding the elements of $\mathrm{Pre}^*(\diamond q_f)$ and $\mathrm{Succ}^*((q_0^+, d_0))$.
- Symbolic graph ensures "globally small" elements
- Refine the notion of symbolic graph at will
- For now: exponential tower in $|Q|$

Summary and Perspectives

- Simple model but still non-trivial model
- Non-atomicity ensures regularity hence decidability

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- Simple model but still non-trivial model
- Non-atomicity ensures regularity hence decidability

- Other properties (is $\diamond q_f = \{(q_f^k, d) \mid k, d\}$ an harder property ?)
- Hardness result with atomic operations ?
- (Local ?) Strategies
- Reasonable PSPACE algorithm
- Modelize concrete problems

Thank you for your attention