A brief introduction to evolutionary game theory

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UMONS

27 October 2015

Outline

The prisoner dilemma

Two suspects are arrested by the police. The police, having separated both prisoners, visit each of them to offer the same deal.

- *If one testifies (Defects) for the prosecution against the other and the other remains silent (Cooperate), the betrayer goes free and the silent accomplice receives the full 10-year sentence.*
- *If both remain silent, both are sentenced to only 3 years in jail.*
- *If each betrays the other, each receives a 5 year sentence.*

How should the prisoners act ?

The prisoner dilemma - the (matrix) game

The matrix associated with the prisoner dilemma :

	C	D
C	$(-3,-3)$	$(-10,0)$
D	$(0,-10)$	$(-5,-5)$

The prisoner dilemma - the (matrix) game

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C	$(-3,-3)$	$(-10,0)$
D	$(0,-10)$	$(-5,-5)$

Equivalently, we can study the matrix below :

	C	D
C	$(3,3)$	$(1,4)$
D	$(4,1)$	$(2,2)$

In both games :

- The action D is strictly dominant.
- The profile (D,D) is the unique Nash equilibrium (even in mixed strategies).

The first point of view : strategic games

	C	D
C	$(3,3)$	$(1,4)$
D	$(4,1)$	$(2,2)$

Rules of the game

- The game is played only once by two players.
- The players choose simultaneously their actions (no communication).
- Each player receives his payoff depending of all the chosen actions.
- The goal of each player is to maximise his own payoff.

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Hypotheses made in strategic games

- The players are **intelligent** *(i.e. they reason perfectly and quickly)*
- The players are **rational** *(i.e. they want to maximise their payoff)*
- The players are **selfish** *(i.e. they only care for their own payoff)*

The first point of view : strategic games

	C	D
C	$(3,3)$	$(1,4)$
D	$(4,1)$	$(2,2)$

(D, D) is the only rational choice !

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The second point of view : infinitely repeated games

	C	D
C	$(3,3)$	$(1,4)$
D	$(4,1)$	$(2,2)$

Rules of the game

- The strategic game is played repeatedly by the same two players.
- The players observe the past moves.
- The payoff is the limit of the average of the payoffs.

Hypotheses made in repeated games

- The players are **intelligent** *(i.e. they reason perfectly and quickly)*
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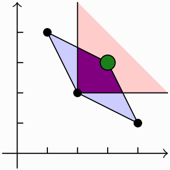

Payoff profiles of rational issues (Folk Theorem)

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The third point of view : evolutionary games

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C	$(3,3)$	$(1,4)$
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We completely change the point of view !

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Rules of the game

- We have a **large** population of individuals.
- Individuals are repeatedly draw at random to play the above game.
- The payoffs are supposed to represent the gain in biological fitness or reproductive value.

The third point of view : evolutionary games

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Hypotheses made in evolutionary games

- Each individual is **genetically programmed** to play either C or D.
- The individuals are no more **intelligent**, nor **rational**, nor **selfish**.

The third point of view : evolutionary games

	C	D
C	$(3,3)$	$(1,4)$
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The strategy D is evolutionary stable, facing an invasion of the mutant strategy C.

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Outline

Strategic games

Definition

A *strategic game* G is a triple $\left(N, (A_i)_{i \in N}, (g_i)_{i \in N}\right)$ where :

- N is the finite and non empty set of players,
- A_k is the non empty set of actions of player k,
- $g_k : \prod_{n \in N} A_k \to \mathbb{R}$ is the payoff function of player k.

	C	D
C	$(3,3)$	$(1,4)$
D	$(4,1)$	$(2,2)$

Domination

Some notations

Given $(E_i)_{i \in N}$ a family of sets :

$$E = \prod_{k \in N} E_k \quad ; \quad E_{-i} = \prod_{k \neq i} E_k .$$

Given $e \in E$:

$$e = (e_1, \dots, e_n) = (e_k)_{k \in N} = (e_i, e_{-i}) \qquad \text{for } i \in N.$$

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Strictly dominated and dominant action (or strategy)

An action $a_i \in A_i$ is **strictly dominated**

$$\exists b_i \in A_i \quad \forall a_{-i} \in A_{-i} \qquad g_i(a_i, a_{-i}) < g_i(b_i, a_{-i})$$

An action $a_i \in A_i$ is **strictly dominant**

$$\forall b_i (\neq a_i) \in A_i \quad \forall a_{-i} \in A_{-i} \qquad g_i(a_i, a_{-i}) > g_i(b_i, a_{-i})$$

Domination - an example

	C	D
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$$\forall b_i (\neq a_i) \in A_i \quad \forall a_{-i} \in A_{-i} \qquad g_i(a_i, a_{-i}) > g_i(b_i, a_{-i})$$

- C is strictly dominated (by D).
- D is strictly dominant.

Nash equilibrium

Nash Equilibrium - Definition

Let (N, A_i, g_i) be a strategic game and $a = (a_i)_{i \in N}$ be a *strategy profile*.

We say that $a = (a_i)_{i \in N}$ is a *Nash equilibrium* iff

$$\forall i \in N \ \forall b_i \in A_i \quad g_i(b_i, a_{-i}) \leqslant g_i(a_i, a_{-i})$$

	C	D
C	$(3, 3)$	$(1, 4)$
D	$(4, 1)$	$(2, 2)$

(D,D) is the unique Nash equilibrium

Mixed strategies

Notations

Given E, we denote $\Delta(E)$ the set of *probability distribution over E*.

Assuming $E = \{e_1, \ldots, e_n\}$, we have that :

$$\Delta(E) = \{(p_1, \ldots, p_n) \mid p_i \geqslant 0 \text{ and } p_1 + \ldots + p_n = 1\}.$$

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Mixed strategy

If A_i is the of strategies of player i, $\Delta(A_i)$ is his set of **mixed strategies**.

Expected payoff

Given $(N, (A_i)_i, (g_i)_i)$. Let $(\sigma_1, \ldots, \sigma_n)$ be a mixed strategies profile.

$$\tilde{g}_i(\sigma_1, \ldots, \sigma_n) = \sum_{a \in A} \underbrace{\left(\prod_{i \in N} \sigma_i(a_i) \right)}_{\text{probability of } a} g_i(a)$$

is the expected payoff of player i.

Nash equilibria in mixed strategies

	L	R
L	$(1, -1)$	$(-1, 1)$
R	$(-1, 1)$	$(1, -1)$

The following profile is a *Nash equilibrium in mixed strategies* :

$$\sigma_1 = \begin{cases} \text{L} & \text{with proba } \frac{1}{2} \\ \text{R} & \text{with proba } \frac{1}{2} \end{cases} \quad \text{and} \quad \sigma_2 = \begin{cases} \text{L} & \text{with proba } \frac{1}{2} \\ \text{R} & \text{with proba } \frac{1}{2} \end{cases}$$

whose expected payoff is 0.

Nash equilibria in mixed strategies

	L	R
L	$(1,-1)$	$(-1,1)$
R	$(-1,1)$	$(1,-1)$

The following profile is a *Nash equilibrium in mixed strategies* :

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whose expected payoff is 0.

Nash Theorem [Nash 1950]

Let G be a finite game.

The game admits mixed Nash equilibria.

Outline

Symmetric two-player game

Symmetric two-player game

A **symmetric two-player game** is a game $(\{1,2\}, A_1, A_2, g_1, g_2)$ where :

- $A_1 = A_2$.
- $\forall (a_1, a_2) \in A_1 \times A_2$, we have that $g_1(a_1, a_2) = g_2(a_2, a_1)$.

Notations : we will denote g_1 by g.

Examples :

	A	B
A	$(1,1)$	$(0,0)$
B	$(0,0)$	$(2,2)$

	C	D
C	$(3,3)$	$(1,4)$
D	$(4,1)$	$(2,2)$

	E	F
E	$(0,0)$	$(3,1)$
F	$(1,3)$	$(2,2)$

Symmetric Nash Equilibrium

Symmetric Nash Equilibrium

A symmetric Nash equilibrium is a Nash equilibrium (σ_1, σ_2) where $\sigma_1 = \sigma_2$.

	A	B
A	$(1,1)$	$(0,0)$
B	$(0,0)$	$(2,2)$

$NE = \{(A, A), (B, B), (\sigma, \sigma)\}$ with $\sigma = \left(\frac{2}{3}, \frac{1}{3}\right)$

	C	D
C	$(3,3)$	$(1,4)$
D	$(4,1)$	$(2,2)$

$NE = \{(D, D)\}$ since D is strictly dominant

	E	F
E	$(0,0)$	$(3,1)$
F	$(1,3)$	$(2,2)$

$NE = \{(E, F), (F, E), (\sigma, \sigma)\}$ with $\sigma = \left(\frac{1}{2}, \frac{1}{2}\right)$

The 2×2 games

	X	Y
X	(a, a)	(b, c)
Y	(c, b)	(d, d)

$\Leftrightarrow$

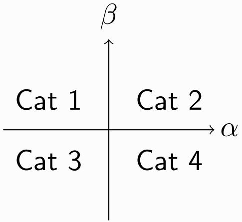
	X	Y
X	$(a - c, a - c)$	$(0, 0)$
Y	$(0, 0)$	$(d - b, d - b)$

$\Leftrightarrow$

	X	Y
X	(α, α)	$(0, 0)$
Y	$(0, 0)$	(β, β)

The 2×2 games - The 4 categories

	X	Y
X	(α, α)	$(0, 0)$
Y	$(0, 0)$	(β, β)



- Cat 1 : $\alpha < 0$ et $\beta > 0$. NE$=\{(Y, Y)\}$
- Cat 2 : α, $\beta > 0$. NE$=\{(X, X), (Y, Y), (\sigma, \sigma)\}$ with $\sigma = \left(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}\right)$
- Cat 3 : α, $\beta < 0$. NE$=\{(X, Y), (Y, X), (\sigma, \sigma)\}$ with $\sigma = \left(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}\right)$
- Cat 4 : $\alpha > 0$ et $\beta < 0$. NE$=\{(X, X)\}$

The generalised Rock-Scissors-Paper Games

	R	P	S
R	$(1,1)$	$(2+a,0)$	$(0,2+a)$
P	$(0,2+a)$	$(1,1)$	$(2+a,0)$
S	$(2+a,0)$	$(0,2+a)$	$(1,1)$

The original RPS game is obtained when $a = 0$.

The generalised RPS game has a unique NE (independently of a) given by (σ, σ), where $\sigma = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

Outline

Evolutionary game theory

We completely change the point of view !

Rules of the game

- We have a **large** population of individuals.
- Individuals are repeatedly draw at random to play a symmetric game.
- The payoffs are supposed to represent the gain in biological fitness or reproductive value.

Hypotheses made in evolutionary games

- Each individual is **genitically programmed** to play a strategy.
- The individuals are no more **intelligent**, nor **rational**, nor **selfish**.

Can an existing population resist to the invasion of a mutant ?

Outline

Evolutionary Stable Strategy

Evolutionary Stable Strategy

We say that σ is an **evolutionary stable strategy (ESS)** iff

- (σ, σ) is a Nash equilibrium.
- For all $\sigma'(\neq \sigma) : g(\sigma', \sigma) = g(\sigma, \sigma) \Rightarrow g(\sigma', \sigma') < g(\sigma, \sigma')$

Thus if (σ, σ) is a **strict** Nash equilibrium, then σ is an ESS.

	A	B
A	$(1, 1)$	$(1, 1)$
B	$(1, 1)$	$(2, 2)$

	C	D
C	$(1, 1)$	$(1, 1)$
D	$(1, 1)$	$(0, 0)$

- (A,A), (B,B) and (C,C) are Nash equilibria.
- A is not an **ESS**.
- B and C are **ESS**.

Evolutionary Stable Strategy - Alternative definition

- Imagine a population composed of a unique species : σ.
- A small proportion ϵ of the population mutes to a “new species” : σ'.
- The new population is thus $\epsilon\sigma' + (1-\epsilon)\sigma$.

Proposition

A strategy σ is an **(ESS)** iff $\forall \sigma' (\neq \sigma)\ \exists \epsilon_0 \in (0,1)\ \forall \epsilon \in (0, \epsilon_0)$

$$g(\sigma, \epsilon\sigma' + (1-\epsilon)\sigma) > g(\sigma', \epsilon\sigma' + (1-\epsilon)\sigma).$$

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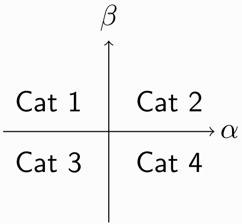
$$g(\sigma, \epsilon\sigma' + (1-\epsilon)\sigma) > g(\sigma', \epsilon\sigma' + (1-\epsilon)\sigma).$$

The concept of **ESS** is a static concept, i.e. it suffices to study the one-shot game.

Evolutionary Stable Strategy - 2×2 games

	X	Y
X	(α, α)	$(0, 0)$
Y	$(0, 0)$	(β, β)



$$
\begin{aligned}
&\textit{Cat } 1 : \text{NE} = \{(Y, Y)\}. && \text{ESS} = \{Y\}. \\
&\textit{Cat } 2 : \text{NE} = \{(X, X), (Y, Y), (\sigma, \sigma)\}. && \text{ESS} = \{X, Y\}. \\
&\textit{Cat } 3 : \text{NE} = \{(X, Y), (Y, X), (\sigma, \sigma)\}. && \text{ESS} = \{\sigma\} \\
&\textit{Cat } 4 : \text{NE} = \{(X, X)\}. && \text{ESS} = \{X\}.
\end{aligned}
$$

Outline

The evolution of a population - intuitively

We have a population composed of several species.

Variation of popu. the species $=$ Popu. of the species $\times$ Advantage of the species

Advantage of the species $=$ Fitness of the species - Average fitness of all species

The evolution of a population - more formally (1)

- We consider a population where individuals are divided into n species. Individuals of species i are programmed to play the pure strategy $\mathbf{a_i}$.
- We denote by $p_i(t)$ the number of individuals of species i at time t.
- The **total population at time** t is given by

$$p(t) = p_1(t) + \cdots + p_n(t).$$

- The **population state at time** t is given by

$$\sigma(t) = (\sigma_1(t), \ldots \sigma_n(t)), \text{ where } \sigma_i(t) = \frac{p_i(t)}{p(t)}.$$

  Note that $\sigma(t) \in \Delta(\{a_1, \ldots, a_n\})$.

The evolution of a population - more formally (2)

The evolution of the state of the population is given by :

The replicator dynamics (RD)

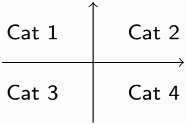
$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma_i(t) = (g(a_i, \sigma(t)) - g(\sigma(t), \sigma(t))) \cdot \sigma_i(t)$$

Theorem

Given any initial condition $\sigma(0) \in \Delta(A)$. The above system of differential equations always admits a unique solution.

The replicator dynamics - 2×2 games

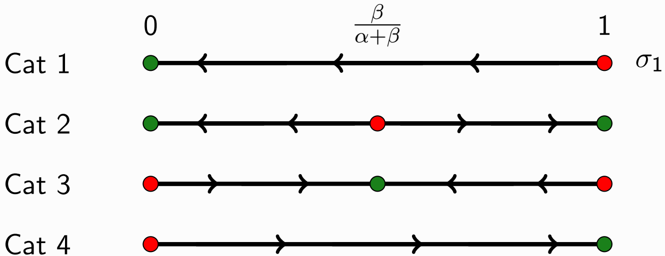
	X	Y
X	(α, α)	$(0,0)$
Y	$(0,0)$	(β, β)



$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\sigma_1(t) = (\alpha\sigma_1(t) - \beta\sigma_2(t)) \cdot \sigma_1(t)\sigma_2(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}\sigma_2(t) = (\beta\sigma_2(t) - \alpha\sigma_1(t)) \cdot \sigma_1(t)\sigma_2(t) \end{cases}$$

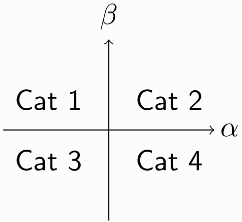
Here $\Delta(A) = \{(\sigma_1, \sigma_2) \in [0,1]^2 \mid \sigma_1 + \sigma_2 = 1\} \simeq [0,1]$, where σ_1 is the prop. of X.

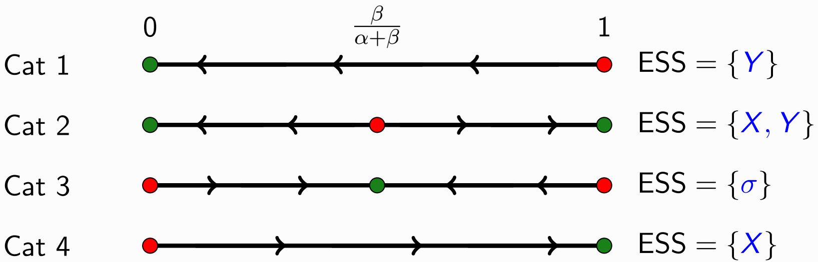
The solutions $(\sigma_1(t), 1 - \sigma_1(t))$ of the (RD) behave as follows :



2×2 games - RD Vs ESS

	X	Y
X	(α, α)	$(0, 0)$
Y	$(0, 0)$	(β, β)



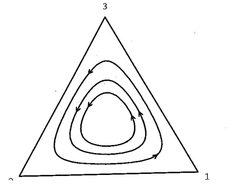


The generalised Rock-Scissors-Paper Games

a=0

$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is not an ESS

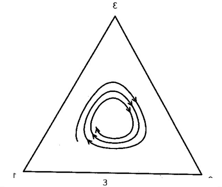
	R	P	S
R	$(1,1)$	$(2,0)$	$(0,2)$
P	$(0,2)$	$(1,1)$	$(2,0)$
S	$(2,0)$	$(0,2)$	$(1,1)$



$a > 0$

$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is an ESS

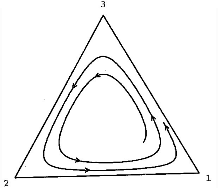
	R	P	S
R	$(1,1)$	$(3,0)$	$(0,3)$
P	$(0,3)$	$(1,1)$	$(3,0)$
S	$(3,0)$	$(0,3)$	$(1,1)$



$a < 0$

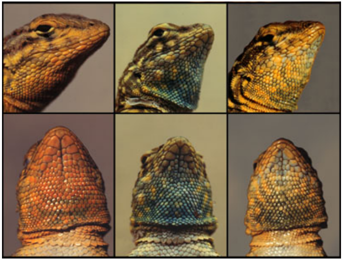
$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is not an ESS

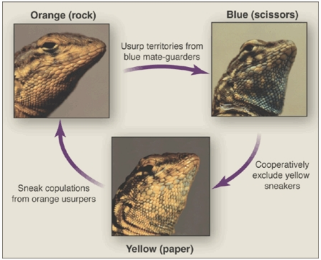
	R	P	S
R	$(1,1)$	$(1,0)$	$(0,1)$
P	$(0,1)$	$(1,1)$	$(1,0)$
S	$(1,0)$	$(0,1)$	$(1,1)$

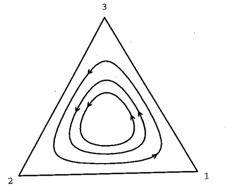


The pictures are taken from *Evolutionnary game theory* by J.W. Weibull.

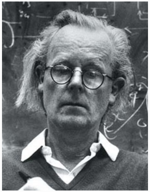
Uta stansburiana - The side-blotched lizard







The populations for these lizards cycle on a six year basis.



When he read that lizards of the species Uta stansburia were essentially engaged in a game with rock-paper-scissors structure John Maynard Smith exclaimed :

They have read my book!

Results

There are several results relating various notions of “static” stability :

- Nash equilibrium,
- Evolutionary Stable Strategy,
- Neutrally Stable Strategy,...

with various notions of “dynamic” stability :

- stationary points,
- Lyapunov stable points,
- asymptotically stable points, ...

Theorems

- If $\sigma \in \Delta$ is Lyapunov stable, then σ is a NE.
- If $\sigma \in \Delta$ is an ESS, then σ is asymptotically stable,...

Outline

Going further...

Rules of the game

- We have a **large** population of individuals.
- Individuals are repeatedly draw at random to play a symmetric game.
- The payoffs are supposed to represent the gain in biological fitness or reproductive value.

Hypotheses made in evolutionary games

- Each individual is **genitically programmed** to play a strategy.
- The individuals are no more **intelligent**, nor **rational**, nor **selfish**.

Replicator dynamics

Variation of popu. the species = Popu. of the species × Advantage of the species

Advantage of the species = Fitness of the species - Average fitness of all species

Going further...

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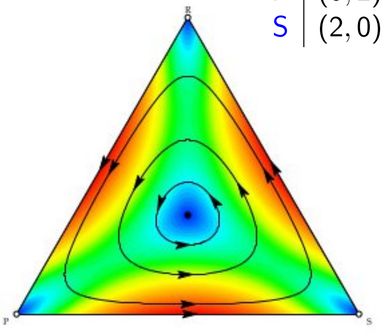
Best-reply dynamics

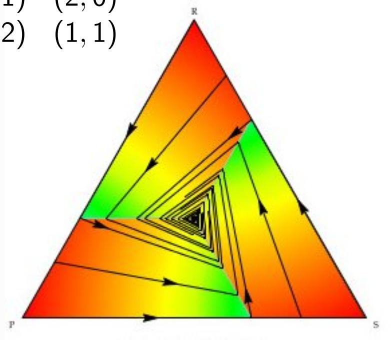
Variation of Strategy Mixture = Best-Reply Strategy - Current Strategy Mixture.

Replicator Vs Best-reply

	R	P	S
R	$(1,1)$	$(2,0)$	$(0,2)$
P	$(0,2)$	$(1,1)$	$(2,0)$
S	$(2,0)$	$(0,2)$	$(1,1)$



Replicator dynamics



Best-reply dynamics

Pictures taken from *Evolutionnary game theory* by W. H. Sandholm

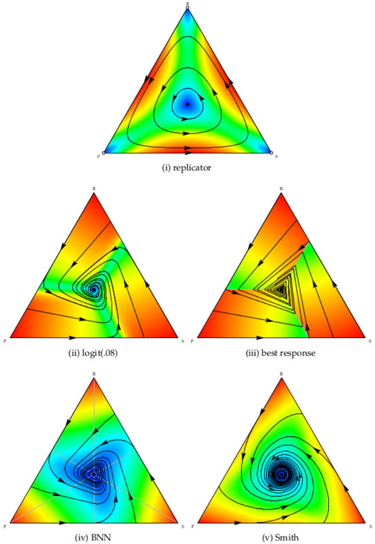


Figure 1: Five basic deterministic dynamics in standard Rock-Paper-Scissors. Colors represent speeds: red is fastest, blue is slowest

Outline

Quotation from Ken Binmore (famous game theorist)

After all, insects can hardly be said to think at all, and so rationality cannot be so crucial if game theory somehow manages to predict their behaviour under appropriate conditions.

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After all, insects can hardly be said to think at all, and so rationality cannot be so crucial if game theory somehow manages to predict their behaviour under appropriate conditions.

Simultaneously the advent of experimental economics brought home the fact that human subjects are no great shakes at thinking either. When they find their way to an equilibrium [...] they typically do so using trial-and-error methods.

Some questions to conclude

During the CASSTING project, we have been interested in

- identifying interesting solution concepts for CAS (winning strategy, various notions of equilibria,...).
- designing efficient algorithm to synthesize these solution concepts.

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To go further, would we benefit from

- a deeper understanding of evolutionnary game theory ?
- studying evolutionnary game theory on games played on graphs ?
- identifying **nicely** converging dynamics for our games ?
    - converging towards which solution concept ? how fast ?
    - could the implementation of the dynamics be less costly than finding directly the solution concept ? For instance, is it less costly to find a best-reply than finding a Nash equilibrium ?

Some books to read

- *Evolutionnary game theory* by J.W. Weibull
- *Evolutionnary game theory* by W. H. Sandholm.
- *Evolutionary Dynamics and Extensive Form Games* by R. Cressman.

Thank you !