About Misère Dicot Games

Gabriel Renault

October 31^{st}, 2014

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- with two players, Left and Right
- with complete information
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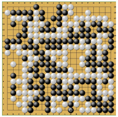
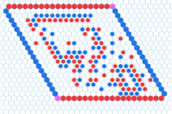

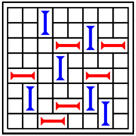

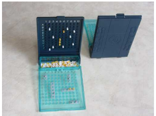

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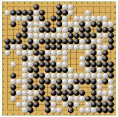
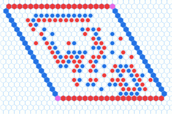

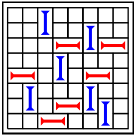



Conventions and outcomes

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There are four possible outcomes for a combinatorial game.

- Left wins $\mathcal{L}$
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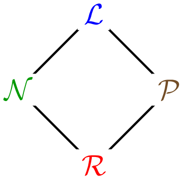

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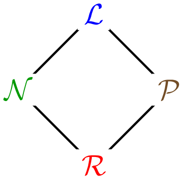
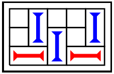


We note $o^+(G)$ the normal outcome of a game G and $o^-(G)$ its misère outcome.

Representation of a game



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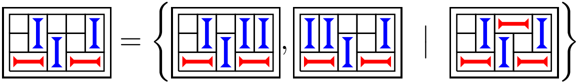

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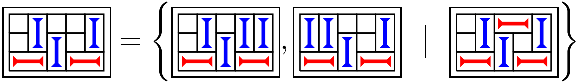

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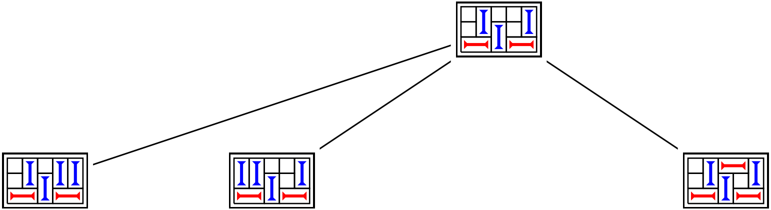

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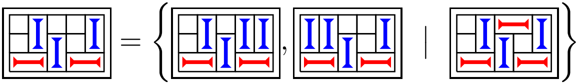

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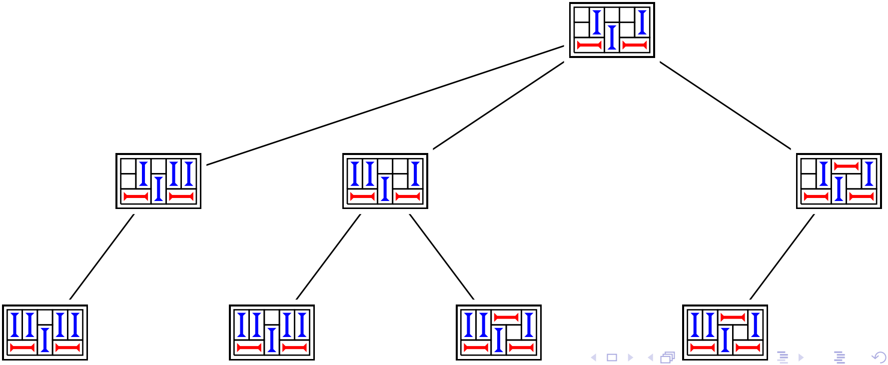

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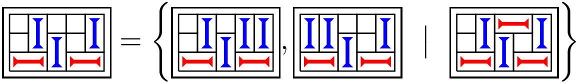

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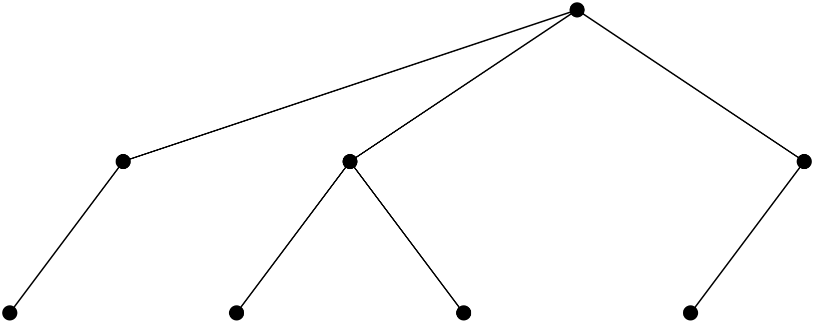

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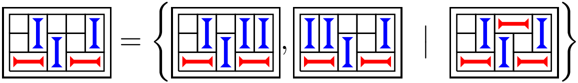

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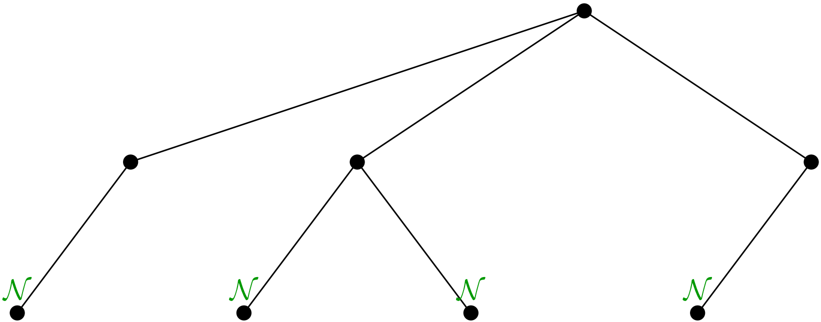

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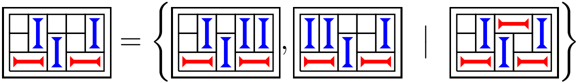

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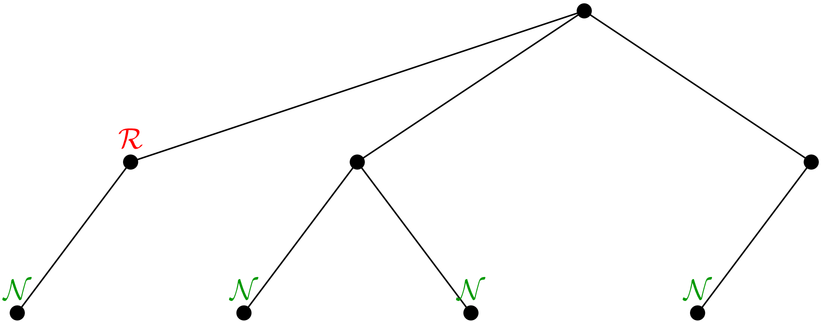

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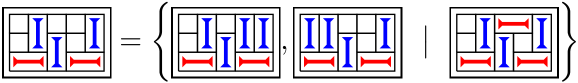

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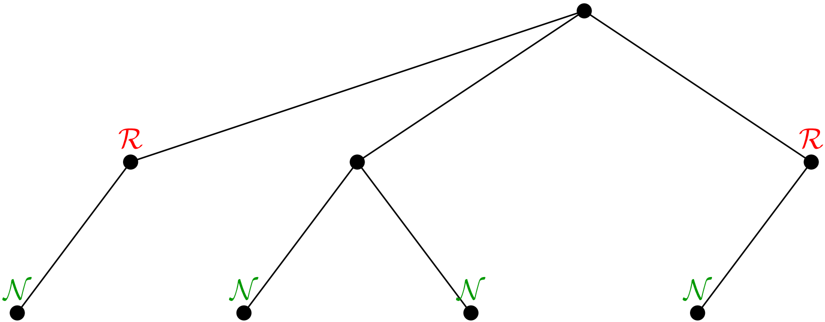

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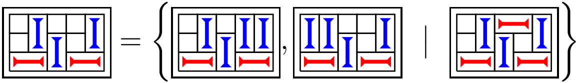

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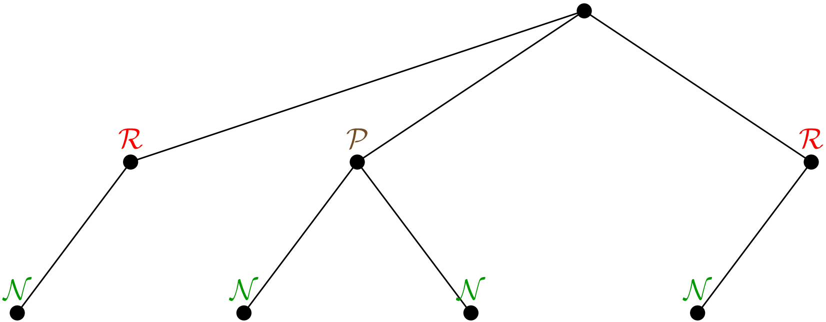

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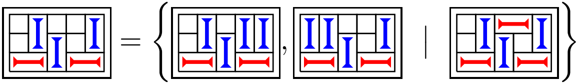

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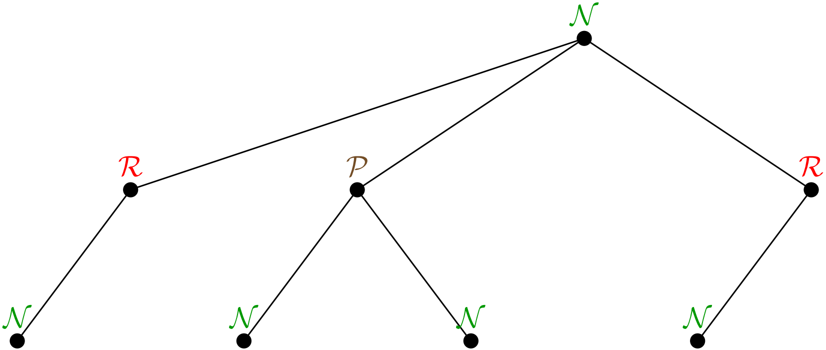
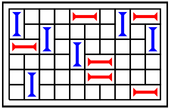
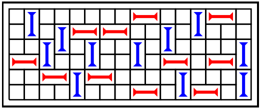


Sum of games

$$G + H = \{G^{L} + H, G + H^{L} | G^{R} + H, G + H^{R}\}$$





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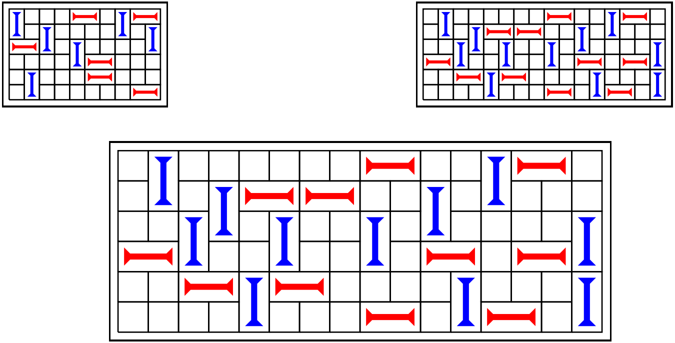

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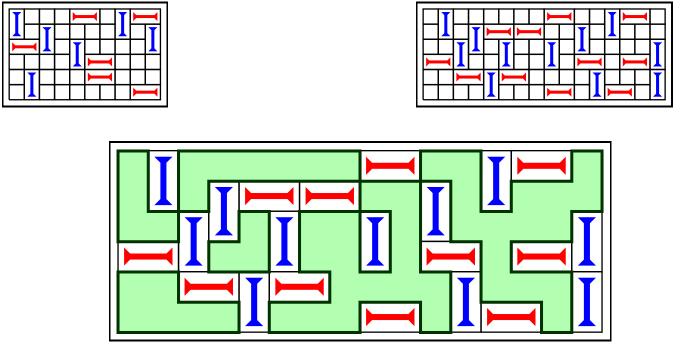

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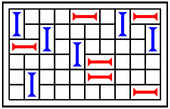

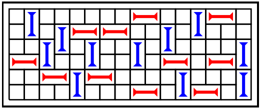

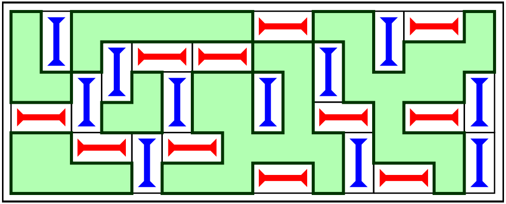

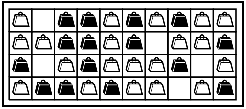

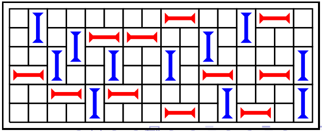

Comparison and equivalence

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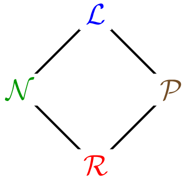

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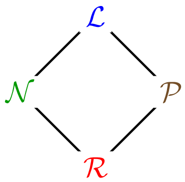

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Theorem (Mesdal, Ottaway):

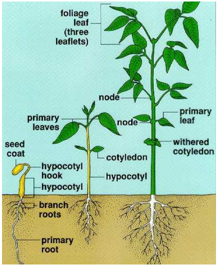
For any game $G \neq 0$, we have $G \not\equiv^- 0$.

Dicot games

Definition:

A game is said to be dicot either if it is $\{\cdot|\cdot\}$ or if it has both Left and Right options and all these options are dicot.

Dicot: $\exists$ Left options $\Leftrightarrow$ $\exists$ Right options.



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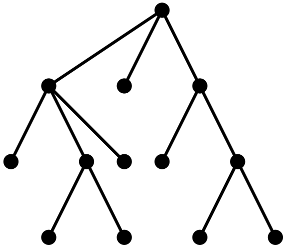

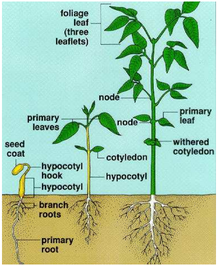

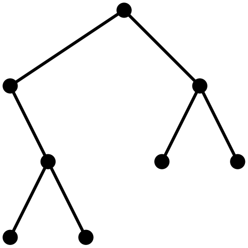

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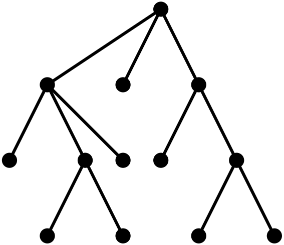

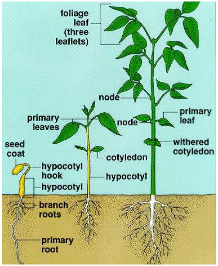

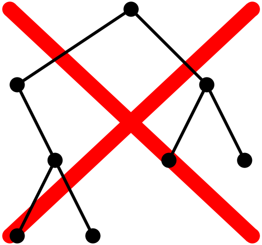

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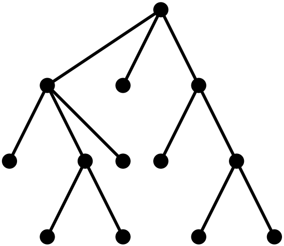

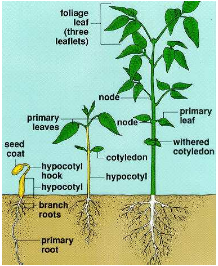

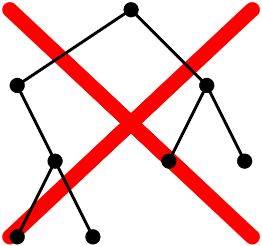

The set of dicot games is closed under sum, follower, and conjugate.
We note $\mathcal{D}$ the set of dicot games.

Comparison and equivalence modulo dicots

$G \geqslant^-_{\mathcal{D}} H$: Left always "prefers" G over H when adding it to a dicot.

Definition (Allen and Nowakowski):

$$(G \geqslant^-_{\mathcal{D}} H) \Leftrightarrow (\forall X \in \mathcal{D}, o^-(G+X) \geqslant o^-(H+X))$$

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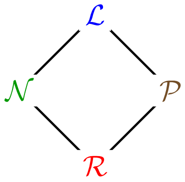

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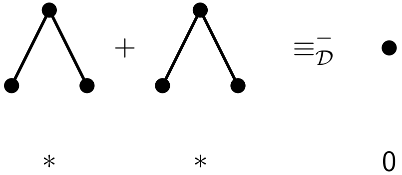

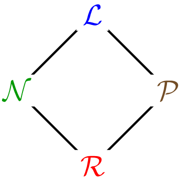

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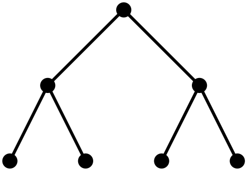

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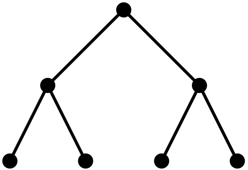

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Misère invertibility

Proposition (Milley):

There exists a set $\mathcal{U}$ and two games G and H in $\mathcal{U}$ such that $G + H \equiv^-_{\mathcal{U}} 0$ and $G + \overline{G} \not\equiv^-_{\mathcal{U}} 0$.

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Question:

For which sets $\mathcal{U}$ do we have $\forall G, H \in \mathcal{U}, (G + H \equiv^-_{\mathcal{U}} 0) \Rightarrow (G + \overline{G} \equiv^-_{\mathcal{U}} 0)$?

Dominated options

If a player always "prefers" an option over another, the second is irrelevant. We say it is dominated.

Definition:

A Left option G^L is $\mathcal{D}$-dominated by some other Left option $G^{L'}$ if $G^{L'} \geqslant^{-}_{\mathcal{D}} G^L$.

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$\mathcal{D}$-dominated options for Right are defined symmetrically.

Lemma:

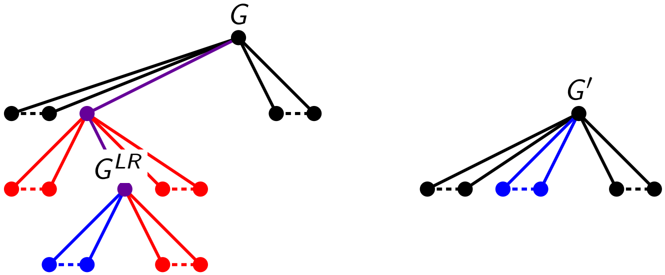
Removing $\mathcal{D}$-dominated options does not change the equivalence class (and in particular the outcome) of a game modulo the dicot universe.

Reversible options

Given a Left option G^L, if Right has a "natural answer" G^{LR}, Left may assume Right plays it and consider G^{LR} options as current options. We say G^L is reversible through G^{LR}.

Definition

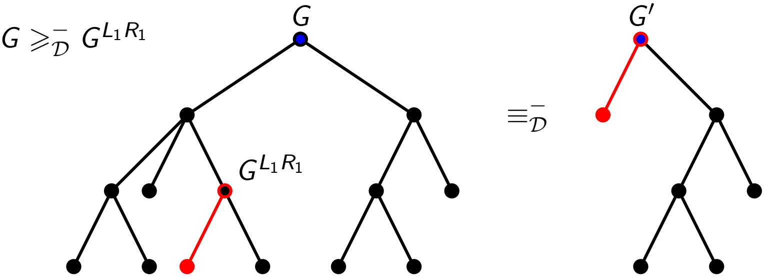
A Left option G^L is $\mathcal{D}$-reversible through some Right option G^{LR} if $G^{LR} \leqslant^-_{\mathcal{D}} G$.



Bypassing reversible options

Lemma:

If G^{L_1} is $\mathcal{D}$-reversible through $G^{L_1R_1}$ and either $G^{L_1R_1} \neq 0$ or there exists another Left option G^{L_2} of G such that $o^-(G^{L_2}) \geqslant \mathcal{P}$, then $G \equiv^-_{\mathcal{D}} G'$ with $G' = \{(G^{L_1R_1})^{\boldsymbol{L}}, G^{\boldsymbol{L}} \setminus \{G^{L_1}\} | G^{\boldsymbol{R}}\}$.



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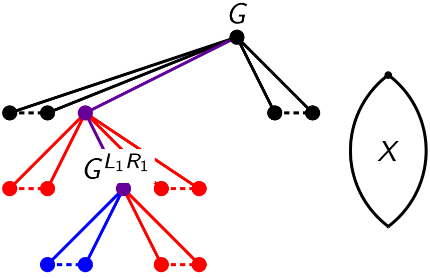
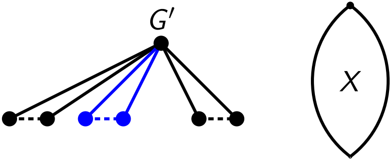
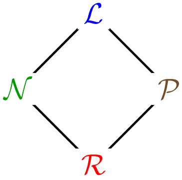


We need $o^-(G+X) = o^-(G'+X)$, that is:

- if Left wins $G+X$, she wins $G'+X$
  $o^-(G+X) \geqslant o^-(G'+X)$
- if Right wins $G+X$, he wins $G'+X$
  $o^-(G+X) \leqslant o^-(G'+X)$





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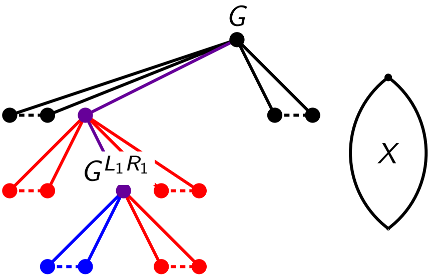
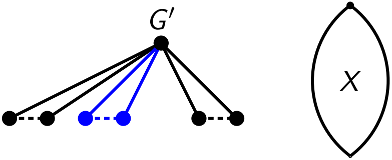
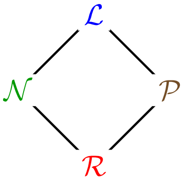


We have:

- $G \geqslant^{-}_{\mathcal{D}} G^{L_1R_1}$
- $G^{L_1R_1} = 0 \Rightarrow$

  $\exists G^{L_2} \in \mathcal{P}^- \cup \mathcal{L}^-$





Reversible options through 0

$\{0, *|*\} \geqslant^-_{\mathcal{D}} 0$, so the Left option $*$ is $\mathcal{D}$-reversible through 0.
$o^-(\{0, *|*\}) = \mathcal{N}$. $o^-(\{0|*\}) = \mathcal{R}$. Hence, $\{0, *|*\} \not\equiv^-_{\mathcal{D}} \{0|*\}$.

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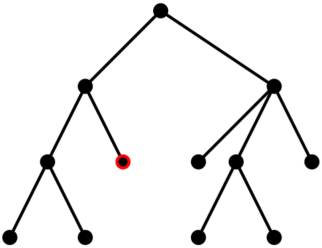
Lemma:

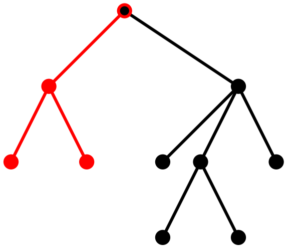
Suppose G^{L_1} is $\mathcal{D}$-reversible through $G^{L_1R_1} = 0$. Let G' be the game obtained by replacing G^{L_1} by $*$:

$$G' = \{*, G^{\boldsymbol{L}} \setminus \{G^{L_1}\} | G^{\boldsymbol{R}}\}.$$

Then G' is a dicot and $G \equiv^-_{\mathcal{D}} G'$.



$\equiv^-_{\mathcal{D}}$



Reduced form of a dicot game

Definition:

Let G be a dicot game. We say G is in *reduced form* if:

(i) it is not $\{*|*\}$,

(ii) it contains no dominated option,

(iii) if Left has a reversible option, it is $*$ and no other Left option has outcome $\mathcal{P}$ or $\mathcal{L}$,

(iv) if Right has a reversible option, it is $*$ and no other Right option has outcome $\mathcal{P}$ or $\mathcal{R}$,

(v) all its options are in reduced form.

Canonical form and comparison with normal play

Lemma:

Let G and H be dicot games such that $G \geqslant^-_{\mathcal{D}} H$. For each Left option H^L of H, there exists:

- either a Left option G^L of G such that $G^L \geqslant^-_{\mathcal{D}} H^L$
- or a Right option H^{LR} of H^L such that $G \geqslant^-_{\mathcal{D}} H^{LR}$.

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Theorem:

Consider two dicot games G and H. If $G \equiv^-_{\mathcal{D}} H$ and both are in reduced form, then $G = H$.

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Theorem:

Let G and H be two dicot games. If $G + H \equiv^-_{\mathcal{D}} 0$, then $G + \overline{G} \equiv^-_{\mathcal{D}} 0$.

Perspectives

- Generalize to sets that share some property
- Detect efficiently comparison between games

Questions?

Thank you.