Control of Nonlinear Switched Systems Based on Validated Simulation

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***Abstract*—We present an algorithm of control synthesis for nonlinear switched systems, based on an existing procedure of state-space bisection and made available for nonlinear systems with the help of validated simulation. The use of validated simulation also permits to take bounded perturbations and varying parameters into account. The whole approach is entirely guaranteed and the induced controllers are *correct-by-design*.**

1. Introduction

We focus here on switched control systems, a class of hybrid systems recently used with success in various domains such as automotive industry and power electronics. These systems are merely described by piecewise dynamics, periodicaly sampled with a given period. At each period, the system is in one and only one mode, decided by a control rule [1], [2].

In this paper, we consider that these modes are represented by nonlinear ODEs. In order to compute the control of a switched system, we do need the solution of differential equations. In the general case, differential equations can not be integrated formally, and a numerical integration scheme is used to approximate the state of the system. With the objective of computing a guaranteed control, we based our approach on validated simulation (also called "reachability analysis"). The *guaranteed* or *validated* solution of ODEs using interval arithmetic is mainly based on two kinds of methods based on: i) Taylor series [3]–[6] ii) Runge-Kutta schemes [7]–[10]. The former is the oldest method used in interval analysis community because the expression of the bound of a Taylor series is simple to obtain. Nevertheless, the family of Runge-Kutta methods is very important in the field of numerical analysis. Indeed, Runge-Kutta methods have several interesting stability properties which make them suitable for an important class of problems. Our tool [11] implements Runge-Kutta based methods which prove their efficiency at low order for short simulation (fixed by sampling period of controller).

In the methods of symbolic analysis and control of hybrid systems, the way of representing sets of state values and computing reachable sets for systems defined by autonomous ordinary differential equations (ODEs), is fundamental (see, e.g., [12], [13]). Many tools using, eg. linearization or hybridization of these dynamics are now available (e.g., Spacex [14], Flow* [15], iSAT-ODE [16]). An interesting approach appeared recently, based on the propagation of reachable sets using guaranteed Runge-Kutta methods with adaptive step size control (see [9], [17]). An originality of the present work is to use such guaranteed integration methods in the framework of switched systems.

The paper is divided as follows. In Section 2, we introduce some preliminaries on switched systems and some notation used in the following. In Section 3, the guaranteed integration of nonlinear ODEs is presented. In Section 4, we present the main algorithm of state-space bisection used for control synthesis. In Section 5, the whole approach is tested on three examples of the literature.

2. Switched systems

Let us consider the nonlinear switched system

$$\dot{x}(t) = f_{\sigma(t)}(x(t), d(t)) \tag{1}$$

defined for all $t \geq 0$, where $x(t) \in \mathbb{R}^n$ is the state of the system, $\sigma(\cdot) : \mathbb{R}^+ \longrightarrow U$ is the switching rule, and $d(t) \in \mathbb{R}^m$ is a bounded perturbation. The finite set $U = \{1, \dots, N\}$ is the set of switching modes of the system. We focus on sampled switched systems: given a sampling period $\tau > 0$, switchings will occur at times τ, 2τ, ... The switching rule $\sigma(\cdot)$ is thus piecewise constant, we will consider that $\sigma(\cdot)$ is constant on the time interval $[(k-1)\tau, k\tau)$ for $k \geq 1$. We call "*pattern*" a finite sequence of modes $\pi = (i_1, i_2, \dots, i_k) \in U^k$. With such a control input, and under a given perturbation d, we will denote by $\mathbf{x}(t; t_0, x_0, d, \pi)$ the solution at time t of the system

$$\begin{aligned} \dot{x}(t) &= f_{\sigma(t)}(x(t), d(t)), \\ x(t_0) &= x_0, \\ \forall j \in \{1, \dots, k\},\ \sigma(t) &= i_j \in U \text{ for } t \in [(j-1)\tau, j\tau). \end{aligned} \tag{2}$$

We address the problem of synthesizing a state-dependent switching rule $\tilde{\sigma}(x)$ for (2) in order to verify some properties. The problem is formalized as follows:

Problem 1. Let us consider a sampled switched system (2). Given three sets R, S, and B, with $R \cup B \subset S$ and $R \cap B = \emptyset$, find a rule $\tilde{\sigma}(x)$ such that, for any $x(0) \in R$

- *τ-stability*[1]: $x(t)$ returns in R infinitely often, at some multiples of sampling time τ.
- *safety*: $x(t)$ always stays in $S \backslash B$.

Under the above-mentioned notation, we propose a procedure which solves this problem by constructing a law $\tilde{\sigma}(x)$, such that for all $x_0 \in R$, and under the unknown bounded perturbation d, there exists $\pi = \tilde{\sigma}(x_0) \in U^k$ for some k such that:

$$\begin{cases} \mathbf{x}(t_0 + k\tau; t_0, x_0, d, \pi) \in R \\ \forall t \in [t_0, t_0 + k\tau], \quad \mathbf{x}(t; t_0, x_0, d, \pi) \in S \\ \forall t \in [t_0, t_0 + k\tau], \quad \mathbf{x}(t; t_0, x_0, d, \pi) \notin B \end{cases}$$

Such a law permits to perform an infinite-time state-dependent control. The synthesis algorithm is described in Section 4 and involves guaranteed set based integration presented in the next section, the main underlying tool is interval analysis [3]. To tackle this problem, we introduce some definitions. In the following, we will often use the notation $[x] \in \mathbb{IR}$ (the set of intervals with real bounds) with $[x] = [\underline{x}, \overline{x}] = \{x \in \mathbb{R} \mid \underline{x} \leqslant x \leqslant \overline{x}\}$ denotes an interval. By an abuse of notation $[x]$ will also denote a vector of intervals, *i.e.*, a Cartesian product of intervals, a.k.a. a *box*. In the following, the sets R, S and B are given under the form of boxes.

Definition 1 (Initial Value Problem (IVP)). Consider an ODE with a given initial condition

$$\dot{x}(t) = f(t, x(t), d(t)) \quad \text{with} \quad x(0) \in X_0,\ d(t) \in [d], \tag{3}$$

with $f : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ assumed to be continuous in t and d and globally Lipschitz in x. We assume that parameters d are bounded (used to represent a perturbation, a modeling error, an uncertainty on measurement, ...). An *IVP* consists in finding a function $x(t)$ described by the ODE (3) for all $d(t)$ lying in $[d]$ and for all the initial conditions in X_0.

Definition 2. Let $X \subset \mathbb{R}^n$ be a box of the state space. Let $\pi = (i_1, i_2, \ldots, i_k) \in U^k$. The *successor set* of X via π, denoted by $Post_\pi(X)$, is the (over-approximation of the) image of X induced by application of the pattern π, *i.e.*, the solution at time $t = k\tau$ of

$$\begin{aligned} \dot{x}(t) &= f_{\sigma(t)}(x(t), d(t)), \\ x(0) &= x_0 \in X, \\ \forall t \geq 0, \quad d(t) &\in [d], \\ \forall j \in \{1, \ldots, k\}, \quad \sigma(t) &= i_j \in U \text{ for } t \in [(j-1)\tau, j\tau). \end{aligned} \tag{4}$$

Definition 3. Let $X \subset \mathbb{R}^n$ be a box of the state space. Let $\pi = (i_1, i_2, \ldots, i_k) \in U^k$. We denote by $Tube_\pi(X)$ the union of boxes covering the trajectories of IVP (4), which construction is detailed in Section 3.

1. This definition of stability is different from the stability in the Lyapunov sense.

3. Validated simulation

In this section, we describe our approach for validated simulation based on Runge-Kutta methods [9], [10].

A numerical integration method computes a sequence of approximations (t_n, x_n) of the solution $x(t; x_0)$ of the IVP defined in Equation (3) such that $x_n \approx x(t_n; x_{n-1})$. The simplest method is Euler's method in which $t_{n+1} = t_n + h$ for some step-size h and $x_{n+1} = x_n + h \times f(t_n, x_n, d)$; so the derivative of x at time t_n, $f(t_n, x_n, d)$, is used as an approximation of the derivative on the whole time interval to perform a linear interpolation. This method is very simple and fast, but requires small step-sizes. More advanced methods coming from the Runge-Kutta family use a few intermediate computations to improve the approximation of the derivative. The general form of an explicit s-stage Runge-Kutta formula, that is using s evaluations of f, is

$$\begin{aligned} x_{n+1} &= x_n + h \sum_{i=1}^{s} b_i k_i \ , \\ k_1 &= f\big(t_n,\, x_n, d\big) \ , \\ k_i &= f\Big(t_n + c_i h,\, x_n + h \sum_{j=1}^{i-1} a_{ij} k_j, d\Big) \ , \quad i = 2, 3, \ldots, s \ . \end{aligned} \tag{5}$$

The coefficients c_i, a_{ij} and b_i fully characterize the method. To make Runge-Kutta validated, the challenging question is how to compute a bound on the distance between the true solution and the numerical solution, defined by $x(t_n; x_{n-1}) - x_n$. This distance is associated to the *local truncation error* (LTE) of the numerical method.

To bound the LTE, we rely on *order condition* [18] respected by all Runge-Kutta methods. This condition states that a method of this family is of order p iff the $p+1$ first coefficients of the Taylor expansion of the solution and the Taylor expansion of the numerical methods are equal. In consequence, LTE is proportional to the Lagrange remainders of Taylor expansions. Formally, LTE is defined by (see [9]):

$$\begin{aligned} x(t_n; x_{n-1}) - x_n = \\ \frac{h^{p+1}}{(p+1)!} \left(f^{(p)}\left(\xi, x(\xi; x_{n-1}), d\right) - \frac{d^{p+1}\phi}{dt^{p+1}}(\eta) \right) \\ \xi \in]t_n, t_{n+1}[\text{ and } \eta \in]t_n, t_{n+1}[\ . \end{aligned} \tag{6}$$

The function $f^{(n)}$ stands for the n-th derivative of function f w.r.t. time t that is $\frac{d^n f}{dt^n}$ and $h = t_{n+1} - t_n$ is the step-size. The function $\phi : \mathbb{R} \to \mathbb{R}^n$ is defined by $\phi(t) = x_n + h \sum_{i=1}^{s} b_i k_i(t)$ where $k_i(t)$ are defined as Equation (5).

The challenge to make Runge-Kutta integration schemes safe w.r.t. the true solution of IVP is then to compute a bound of the result of Equation (6). In other words we have to bound the value of $f^{(p)}\left(\xi, x(\xi; x_{n-1}), d\right)$ and the value of $\frac{d^{p+1}\phi}{dt^{p+1}}(\eta)$. The latter expression is straightforward to bound because the function ϕ only depends on the value of the step-size h, and so does its $(p+1)$-th derivative. The bound is then obtain using the affine arithmetic.

However, the expression $f^{(p)}(\xi, x(\xi; x_{n-1}), d)$ is not so easy to bound as it requires to evaluate f for a particular value of the IVP solution $x(\xi; x_{n-1})$ at an unknown time $\xi \in]t_n, t_{n+1}[$. The solution used is the same as the one found in [4], [7] and it requires to bound the solution of IVP on the interval $[t_n, t_{n+1}]$. This bound is usually computed using the Banach's fixpoint theorem applied with the Picard-Lindelöf operator, see [4]. This operator is used to compute an enclosure of the solution $[\tilde{x}]$ of IVP over a time interval $[t_n, t_{n+1}]$, that is for all $t \in [t_n, t_{n+1}]$, $x(t; x_{n-1}) \in [\tilde{x}]$. We can hence bound $f^{(p)}$ substituting $x(\xi; x_{n-1})$ by $[\tilde{x}]$.

For a given pattern of switched modes $\pi = (i_1, i_2, \ldots, i_k) \in U^k$ of length k, we are able to compute, for $j \in \{1, .., k\}$, the enclosures:

- $[x_j] \ni x(t_j)$;
- $[\tilde{x}_j] \ni x(t)$, for $t \in [(j-1)\tau, j\tau]$.

with respect to the system of IVPs:

$$\begin{cases} \dot{x}(t) = f_{\sigma(t)}(t, x(t), d(t)), \\ x(t_0 = 0) \in [x_0], d(t) \in [d], \\ \sigma(t) = i_1, \forall t \in [0, t_1], t_1 = \tau \\ \vdots \\ \dot{x}(t) = f_{\sigma(t)}(t, x(t), d(t)), \\ x(t_{k-1}) \in [x_{k-1}], d(t) \in [d], \\ \sigma(t) = i_k, \forall t \in [t_{k-1}, t_k], t_k = k\tau \end{cases}$$

Thereby, the enclosure $Post_\pi([x_0])$ is included in $[x_k]$ and $Tube_\pi([x_0])$ is included in $\bigcup_{j=1,\ldots,k} [\tilde{x}_j]$. This applies for all initial states in $[x_0]$ and all disturbances $d(t) \in [d]$. A view of enclosures computed by the validated simulation for one solution obtained for Example 5.2 is shown in Figure 1.

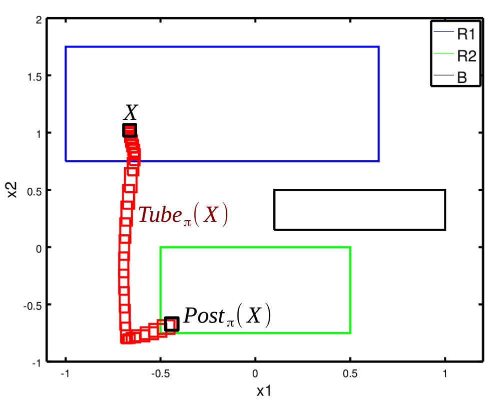

Figure 1. Functions $Post_\pi(X)$ and $Tube_\pi(X)$ for the initial box $X = ([-0.69, -0.64] \times [1, 1.06])$, with a pattern $\pi = (1, 3, 0)$.

4. The state-space bisection algorithm

We describe here the main algorithm of control synthesis. Given the input boxes R, S, B, and given two positive integers K and D, the algorithm provides, when it succeeds, a decomposition Δ of R of the form $\{V_i, \pi_i\}_{i \in I}$, with the properties:

$\bigcup_{i \in I} V_i = R$
$\forall i \in I,\ Post_{\pi_i}(V_i) \subseteq R$
$\forall i \in I,\ Tube_{\pi_i}(V_i) \subseteq S$
$\forall i \in I,\ Tube_{\pi_i}(V_i) \bigcap B = \emptyset$

The sub-boxes $\{V_i\}_{i \in I}$ are obtained by repeated bisection. At first, function $Decomposition$ calls sub-function $Find\_Pattern$ which looks for a pattern π of length at most K such that $Post_\pi(R) \subseteq R$, $Tube_\pi(R) \subseteq S$ and $Tube_\pi(R) \bigcap B = \emptyset$. If such a pattern π is found, then a uniform control over R is found (see Figure 2(a)). Otherwise, R is divided into two sub-boxes V_1, V_2, by bisecting R w.r.t. its longest dimension. Patterns are then searched to control these sub-boxes (see Figure 2(b)). If for each V_i, function $Find\_Pattern$ manages to get a pattern π_i of length at most K verifying $Post_{\pi_i}(V_i) \subseteq R$, $Tube_{\pi_i}(V_i) \subseteq S$ and $Tube_{\pi_i}(V_i) \bigcap B = \emptyset$, then it is done. If, for some V_j, no such pattern is found, the procedure is recursively applied to V_j. It ends with success when every sub-box of R has a pattern verifying the latter conditions, or fails when the maximal degree of decomposition D is reached. The algorithmic form of functions $Decomposition$ and $Find\_Pattern$ is given in [1] for the linear case.

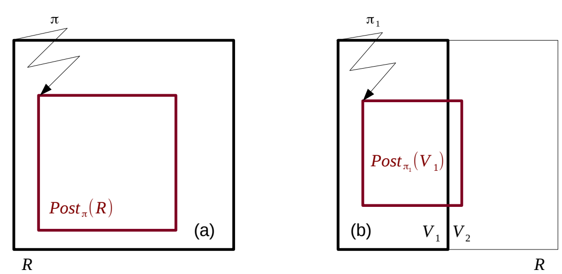

Figure 2. Principle of the bisection method.

Having defined the control synthesis method, we now introduce the main result of this paper, stated as follows:

Proposition 1. A decomposition successfully computed with the above procedure allows to perform an infinite-time state-dependent control satisfying Problem 1.

Proof: Let $x_0 = x(t_0 = 0)$ be an initial condition belonging to R. If the decomposition has terminated successfully, we have $\bigcup_{i \in I} V_i = R$, and x_0 thus belongs to V_{i_0} for some $i_0 \in I$. We can thus apply the pattern π_{i_0} associated to V_{i_0}. Let us denote by k_0 the length of π_{i_0}. We have:

- $\mathbf{x}(k_0\tau; 0, x_0, d, \pi_{i_0}) \in R$
- $\forall t \in [0, k_0\tau]$, $\quad \mathbf{x}(t; 0, x_0, d, \pi_{i_0}) \in S$
- $\forall t \in [0, k_0\tau]$, $\quad \mathbf{x}(t; 0, x_0, d, \pi_{i_0}) \notin B$

Let $x_1 = \mathbf{x}(k_0\tau; 0, x_0, d, \pi_{i_0}) \in R$ be the state reached after application of π_{i_0} and let $t_1 = k_0\tau$. State x_1 belongs to R, it thus belongs to V_{i_1} for some $i_1 \in I$, and we can apply the associated pattern π_{i_1} of length k_1, leading to:

- $\mathbf{x}(t_1 + k_1\tau; t_1, x_1, d, \pi_{i_1}) \in R$

- $\forall t \in [t_1, t_1 + k_1\tau], \quad \mathbf{x}(t; t_1, x_1, d, \pi_{i_1}) \in S$
- $\forall t \in [t_1, t_1 + k_1\tau], \quad \mathbf{x}(t; t_1, x_1, d, \pi_{i_1}) \notin B$

We can then iterate this procedure from the new state $x_2 = \mathbf{x}(t_1 + k_1\tau; t_1, x_1, d, \pi_{i_1}) \in R$. This can be repeated infinitely, yielding a sequence of points belonging to R $x_0, x_1, x_2, \ldots$ attained at times $t_0, t_1, t_2, \ldots$, at which the patterns $\pi_{i_0}, \pi_{i_1}, \pi_{i_2}, \ldots$ are applied.

We furthermore have that all the trajectories stay in S and never cross B: $\forall t \in \mathbb{R}^+, \exists k \geq 0,\ t \in [t_k, t_{k+1}]$ and $\forall t \in [t_k, t_{k+1}],\ \mathbf{x}(t; t_k, x_k, d, \pi_{i_k}) \in S,\ \mathbf{x}(t; t_k, x_k, d, \pi_{i_k}) \notin B$. The trajectories thus return infinitely often in R, while always staying in S and never crossing B. $\square$

Remark 1. Note that it is possible to perform reachability from a set R_1 to another set R_2 by computing $Decomposition(R_1, R_2, S, B, D, K)$. The set R_1 is thus decomposed with the objective to send its sub-boxes into R_2, i.e. for a sub-box V of R_1, patterns π are searched with the objective $Post_\pi(V) \subseteq R_2$ (see Example 5.2).

Remark 2. Our solver prototype, written in C++ and based on DynIBEX [11], also exploits heuristics to prune the search tree of patterns. For example, cross the set B, then all the branches issued from it will also cross B, and this branch should thus be cut.

5. Experimentations

In this section, we apply our approach to different case studies taken from the literature. The computations times given in the following have been performed on a 2.80 GHz Intel Core i7-4810MQ CPU with 8 GB of memory. Note that our algorithm is mono-threaded so all the experimentation only uses one core to perform the computations.

5.1. Boost DC-DC converter

This linear example is taken from [19] and has already been treated with the state-space bisection method in a linear framework in [1].

The system is a boost DC-DC converter with one switching cell. There are two switching modes depending on the position of the switching cell. The dynamics is given by the equation $\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}$ with $\sigma(t) \in U = \{1, 2\}$. The two modes are given by the matrices:

$$A_1 = \begin{pmatrix} -\frac{r_l}{x_l} & 0 \\ 0 & -\frac{1}{x_c}\frac{1}{r_0+r_c} \end{pmatrix} \quad B_1 = \begin{pmatrix} \frac{v_s}{x_l} \\ 0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} -\frac{1}{x_l}(r_l + \frac{r_0.r_c}{r_0+r_c}) & -\frac{1}{x_l}\frac{r_0}{r_0+r_c} \\ \frac{1}{x_c}\frac{r_0}{r_0+r_c} & -\frac{1}{x_c}\frac{r_0}{r_0+r_c} \end{pmatrix} \quad B_2 = \begin{pmatrix} \frac{v_s}{x_l} \\ 0 \end{pmatrix}$$

with $x_c = 70$, $x_l = 3$, $r_c = 0.005$, $r_l = 0.05$, $r_0 = 1$, $v_s = 1$. The sampling period is $\tau = 0.5$. The parameters are exact and there is no perturbation. We want the state to return infinitely often to the region R, set here to $[1.55, 2.15] \times [1.0, 1.4]$, while never going out of the safety set $S = [1.54, 2.16] \times [0.99, 1.41]$.

The decomposition was obtained in less than one second with a maximum length of pattern set to $K = 6$ and a maximum bisection depth of $D = 3$. A simulation is given in Figure 3.

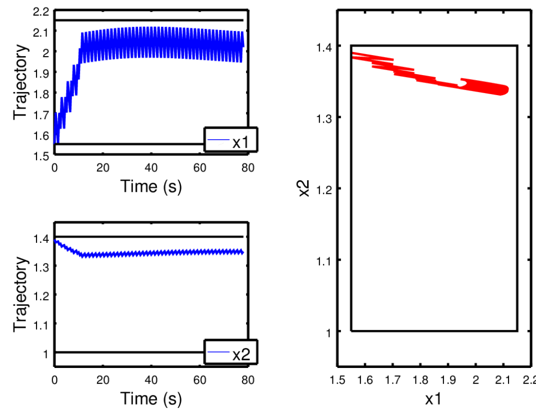

Figure 3. Simulation from the initial condition $(1.55, 1.4)$. The box R is in plain black. The trajectory is plotted within time for the two state variables on the left, and in the state-space plane on the right.

5.2. A polynomial example

We consider the polynomial system taken from [20]:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_2 - 1.5x_1 - 0.5x_1^3 + u_1 + d_1 \\ x_1 + u_2 + d_2 \end{bmatrix}. \quad (7)$$

The control inputs are given by $u = (u_1, u_2) = K_{\sigma(t)}(x_1, x_2)$, $\sigma(t) \in U = \{1, 2, 3, 4\}$, which are four different state feedback controllers $K_1(x) = (0, -x_2^2 + 2)$, $K_2(x) = (0, -x_2)$, $K_3(x) = (2, 10)$, $K_4(x) = (-1.5, 10)$. We thus have four switching modes. The disturbance $d = (d_1, d_2)$ lies in $[-0.005, 0.005] \times [-0.005, 0.005]$. The objective is to visit infinitely often two zones R_1 and R_2, without going out of a safety zone S, and while never crossing a forbidden zone B. Two decompositions are performed:

- a decomposition of R_1 which returns $\{(V_i, \pi_i)\}_{i \in I_1}$ with:
  $\bigcup_{i \in I_1} V_i = R_1$,
  $\forall i \in I_1,\ Post_{\pi_i}(V_i) \subseteq R_2$,
  $\forall i \in I_1,\ Tube_{\pi_i}(V_i) \subseteq S$,
  $\forall i \in I_1,\ Tube_{\pi_i}(V_i) \bigcap B = \emptyset$.
- a decomposition of R_2 which returns $\{(V_i, \pi_i)\}_{i \in I_2}$ with:
  $\bigcup_{i \in I_2} V_i = R_2$,
  $\forall i \in I_2,\ Post_{\pi_i}(V_i) \subseteq R_1$,
  $\forall i \in I_2,\ Tube_{\pi_i}(V_i) \subseteq S$,
  $\forall i \in I_2,\ Tube_{\pi_i}(V_i) \bigcap B = \emptyset$.

The input boxes are the following:

$R_1 = [-0.5, 0.5] \times [-0.75, 0.0]$,
$R_2 = [-1.0, 0.65] \times [0.75, 1.75]$,
$S = [-2.0, 2.0] \times [-1.5, 3.0]$,
$B = [0.1, 1.0] \times [0.15, 0.5]$.

The sampling period is set to $\tau = 0.15$. The decompositions were obtained in 2 minutes and 30 seconds with a maximum length of pattern set to $K = 12$ and a maximum bisection depth of $D = 5$. A simulation is given in Figure 4 in which the disturbance d is chosen randomly in $[-0.005, 0.005] \times [-0.005, 0.005]$ at every time step.

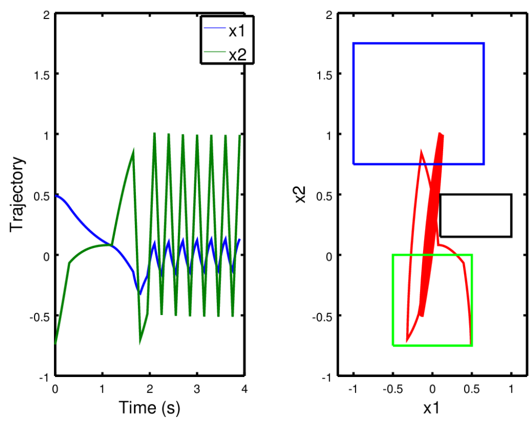

Figure 4. Simulation from the initial condition $(0.5, -0.75)$. The trajectory is plotted within time on the left, and in the state space plane on the right. In the sate space plane, the set R_1 is in plain green, R_2 in plain blue, and B in plain black.

5.3. Building ventilation

We consider a building ventilation application adapted from [21]. The system is a four room apartment subject to heat transfer between the rooms, with the external environment, with the underfloor, and with human beings. The dynamics of the system is given by the following equation:

$$\frac{dT_i}{dt} = \sum_{j \in \mathcal{N}^*} a_{ij}(T_j - T_i) + \delta_{s_i} b_i (T_{s_i}^4 - T_i^4) + c_i \max\left(0, \frac{V_i - V_i^*}{\bar{V}_i - V_i^*}\right)(T_u - T_i). \quad (8)$$

The state of the system is given by the temperatures in the rooms T_i, for $i \in \mathcal{N} = \{1, \ldots, 4\}$. Room i is subject to heat exchange with different entities stated by the indexes $\mathcal{N}^* = \{1, 2, 3, 4, u, o, c\}$.

The heat transfer between the rooms is given by the coefficients a_{ij} for $i, j \in \mathcal{N}^2$, and the different perturbations are the following:

- The external environment: it has an effect on room i with the coefficient a_{io} and the outside temperature T_o, varying between $27°C$ and $30°C$.
- The heat transfer through the ceiling: it has an effect on room i with the coefficient a_{ic} and the ceiling temperature T_c, varying between $27°C$ and $30°C$.
- The heat transfer with the underfloor: it is given by the coefficient a_{iu} and the underfloor temperature T_u, set to $17°C$ (T_u is constant, regulated by a PID controller).
- The perturbation induced by the presence of humans: it is given in room i by the term $\delta_{s_i} b_i (T_{s_i}^4 - T_i^4)$, the parameter δ_{s_i} is equal to 1 when someone is present in room i, 0 otherwise, and T_{s_i} is a given identified parameter.

The control V_i, $i \in \mathcal{N}$, is applied through the term $c_i \max(0, \frac{V_i - V_i^*}{\bar{V}_i - V_i^*})(T_u - T_i)$. A voltage V_i is applied to force ventilation from the underfloor to room i, and the command of an underfloor fan is subject to a dry friction. Because we work in a switched control framework, V_i can take only discrete values, which removes the problem of dealing with a "max" function in interval analysis. In the experiment, V_1 and V_4 can take the values 0V or 3.5V, and V_2 and V_3 can take the values 0V or 3V. This leads to a system of the form (1) with $\sigma(t) \in U = \{1, \ldots, 16\}$, the 16 switching modes corresponding to the different possible combinations of voltages V_i. The sampling period is $\tau = 10$s.

The parameters T_{s_i}, V_i^*, $\bar{V}_i$, a_{ij}, b_i, c_i are given in [21] and have been identified with a proper identification procedure detailed in [22]. Note that we have neglected the term $\sum_{j \in \mathcal{N}} \delta_{d_{ij}} c_{i,j} * h(T_j - T_i)$ of [21], representing the perturbation induced by the open or closed state of the doors between the rooms. Taking a "max" function into account with interval analysis is actually still a difficult task. However, this term could have been taken into account with a proper regularization (smoothing).

The decomposition was obtained in 4 minutes with a maximum length of pattern set to $K = 2$ and a maximum bisection depth of $D = 4$. The perturbation due to human beings has been taken into account by setting the parameters δ_{s_i} equal to the whole interval $[0, 1]$ for the decomposition, and the imposed perturbation for the simulation is given Figure 5. The temperatures T_o and T_c have been set to the interval $[27, 30]$ for the decomposition, and are set to $30°C$ for the simulation. A simulation of the controller obtained with the state-space bisection procedure is given in Figure 6, where the control objective is to stabilize the temperature in $[20, 22]^4$ while never going out of $[19, 23]^4$.

6. Conclusion

We presented a method of control synthesis for nonlinear switched systems, based on a simple state-space bisection algorithm, and on validated simulation. The approach permits to deal with stability, safety and forbidden region constraints. Varying parameters and perturbations can be easily taken into account with interval analysis. The approach has been numerically validated on three examples taken from the literature, a linear one with constant parameters, and two nonlinear ones with varying perturbations.

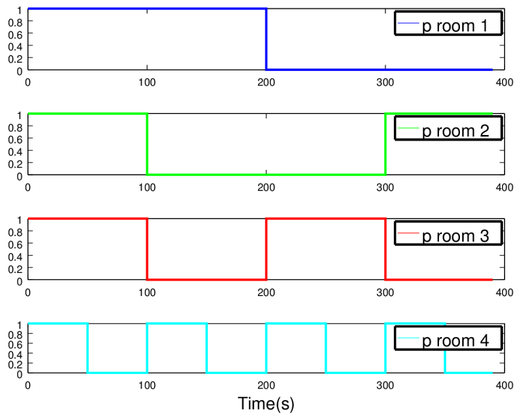

Figure 5. Perturbation (presence of humans) imposed within time in the different rooms.

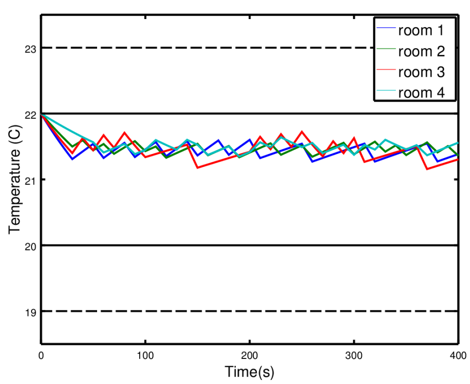

Figure 6. Simulation from the initial condition $(22, 22, 22, 22)$. The objective set R is in plain black and the safety set S is in dotted black.

We are currently continuing the improvements mentioned in Remark 2, in order to be able to handle longer patterns and higher dynamics dimension. Our future work will be also devoted to the extension of this method to the control of nonlinear partial differential equations.

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