On the complexity of heterogeneous multidimensional quantitative games

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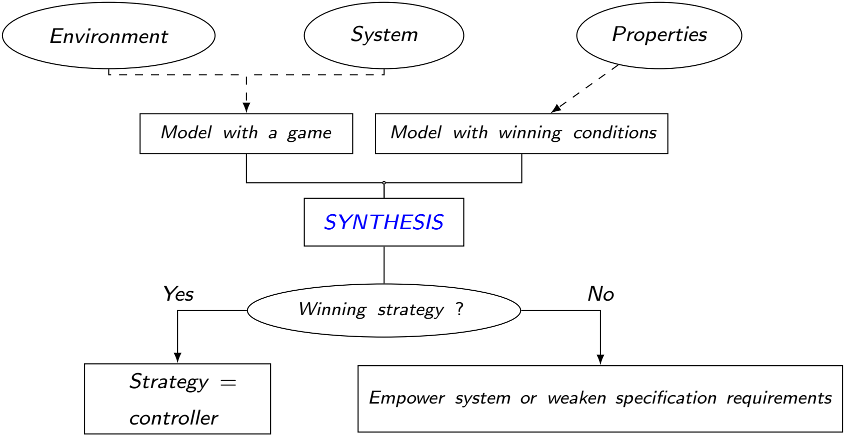
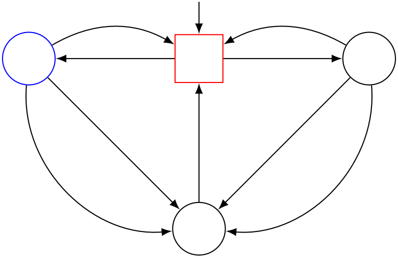
Synthesis via Game Theory



Model

Zero-sum games played on finite graph:

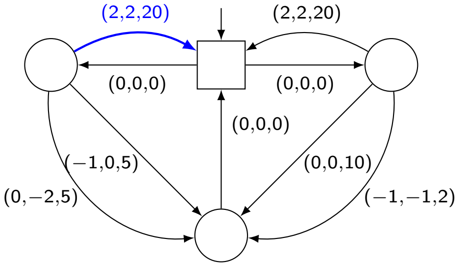
- System vs. Environment : antagonistic
- Turn-based games



Model

Multi-dimensional weighted zero-sum games played on finite graph:

- Weight for energy, time consumption, ...



Pure strategies $\sigma_p : V^* V_p \to V$.

Known results

- Uni-dimensional ([Jur98] [CDRR15])

	Inf	LimInf	Sup	LimSup	MP	En.	WMP
Complexity	P-complete				NP $\cap$ coNP		P-c
P1 memory	memoryless						exponential
P2 memory	memoryless						exponential

- multi-dimensional: Homogeneous intersection ([CDHR10] [CDRR15])

	En	$\underline{\text{MP}}$	$\overline{\text{MP}}$	WMP
Complexity	coNP-c		NP $\cap$ coNP	EXPTIME-c
P1 memory	finite-memory	infinite-memory		exponential
P2 memory	memoryless			exponential

- Boolean combinations of $\overline{\text{MP}}$ and $\underline{\text{MP}}$: undecidable. [Vel15]

Problem

We consider here heterogeneous objectives.

- One objective by dimension
- Objective : measure of the play $\geq$ threshold ν

Definition

Let $G = (V_1, V_2, E, w)$ be a n-weighted game structure, $\nu = (\nu_1, \ldots, \nu_n)$ be a threshold vector and $\Omega = \bigcap_{m=1}^{n} \Omega_m$ such that Ω_m is a quantitative objective regarding ν_m. The *threshold problem* asks to decide whether player 1 has a winning strategy for Ω from an initial vertex v_0.

Quantitative measures studied[1]:

- Inf (Sup) : minimum (maximum) weight seen
- LimInf (LimSup): minimum (maximum) weight infinitely seen
- *WindowMeanPayoff* (WMP): average weight over a local window sliding along the play

Definition

Given a threshold $\nu \in \mathbb{Q}$ and a window size $\lambda \in \mathbb{N} \setminus \{0\}$,

$$\mathsf{WMP}(\lambda, \nu) = \{\rho \in \mathsf{Plays}(G) \mid \forall k \geq 0, \exists l \in \{1, \ldots, \lambda\}, \mathsf{MP}(\rho_{[k,k+l]}) \geq \nu\}.$$

[1] All those measures are ω-regular

WMP(λ, 0)

Window at position k is

- closed in $k + l$ if $\exists l \in \{1, \ldots, \lambda\}$ s.t. $\mathrm{MP}(\rho_{[k,k+l]}) \geq 0$,

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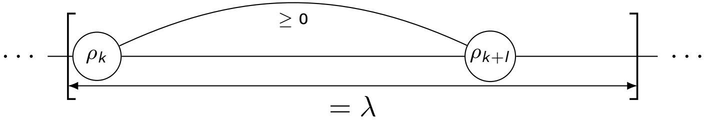

WMP($\lambda, 0$)

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- first-closed in $k + l^*$ if l^* is minimal,

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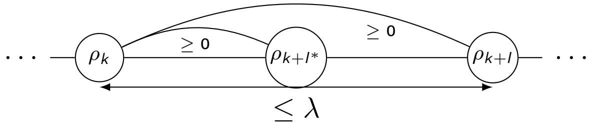

WMP$(\lambda, 0)$

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- inductively-closed in $k + l$ if it closed in $k + l$ and this is also the case for each $k' \in \{k + 1, \ldots, k + l\}$.

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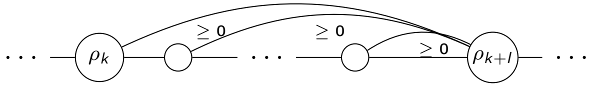

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- inductively-closed in $k + l$ if it closed in $k + l$ and this is also the case for each $k' \in \{k + 1, \ldots, k + l\}$.

A window that is first-closed is inductively-closed.

Lemma

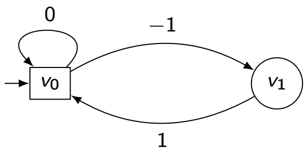
$\rho \in \mathsf{WMP}(\lambda, 0)$ *iff there exists a sequence* $(k_i)_i$ *such that,* $k_0 = \rho_0$ *and for each* i*,* $k_{i+1} - k_i \leq \lambda$ *and the window at position* k_i *is inductively-closed in* k_{i+1}*.*

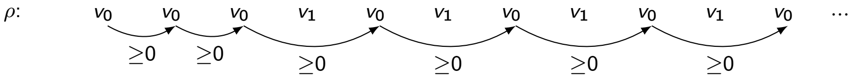
Lemma

$\rho \in \mathrm{WMP}(\lambda, 0)$ iff there exists a sequence $(k_i)_i$ such that, $k_0 = \rho_0$ and for each i, $k_{i+1} - k_i \leq \lambda$ and the window at position k_i is inductively-closed in k_{i+1}.

Example:

WMP(2,0):





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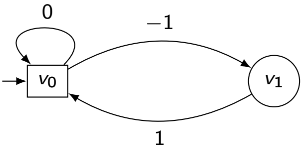
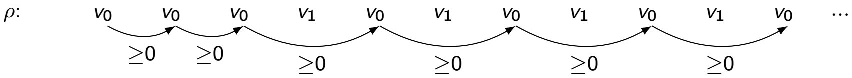




$k_0 = 0,\ k_1 = 1,\ k_2 = 2,\ i > 2,\ k_i = k_{i-1} + 2$

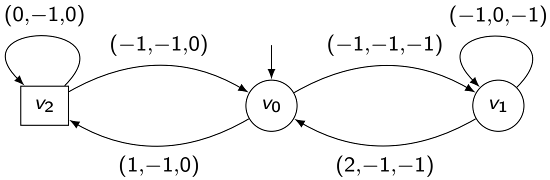
Questions

Threshold problem for $\Omega = \bigcap_{m=1}^{n} \Omega_m$ with $\Omega_m \in \{$WMP, Inf, Sup, LimInf, LimSup$\}$.

- Is the threshold problem decidable ?
- If yes, what is the complexity class ?
- What kind of strategies do the players need ?

Example

$\Omega = \mathrm{WMP}(3,0) \cap \mathrm{Sup}(0) \cap \mathrm{LimSup}(0)$



$\sigma_1(v_0) = v_1$, $\sigma_1(v_0v_1) = v_1$, $\sigma_1(v_0v_1v_1) = v_0$, and $\sigma_1(v_0v_1v_1hv_0) = v_2$ for all $hv_0 \in V^*V_1$

Results

Objectives	Complexity class	Player 1 memory	Player 2 memory
Intersection of WMP, Inf, Sup, LimInf, LimSup	EXPTIME-complete	exponential	
Intersection of Inf, Sup, LimInf, LimSup and refinements	PSPACE-complete	exponential	
	Table of section "PSPACE fragment"		
WMP $\cap\ \Omega$ with $\Omega \in \{$Inf, Sup, LimInf, LimSup$\}$	P-complete	exponential	

General result

<table>
<tr><th>Objectives</th><th>Complexity class</th><th>Player 1 memory</th><th>Player 2 memory</th></tr>
<tr><td>Intersection of WMP, Inf, Sup, LimInf, LimSup</td><td>EXPTIME-complete</td><td colspan="2" rowspan="2">exponential</td></tr>
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and refinements</td><td>PSPACE-complete</td></tr>
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</table>

General result

Theorem

Let (G, Ω) be an n-weighted game such that $\Omega = \cap_{m=1}^{n} \Omega_m$ with $\Omega_m \in \{\text{WMP, Inf, Sup, LimInf, LimSup}\}$ for all m. Then, the threshold problem is EXPTIME-complete (with an algorithm in time $O(|V| \cdot |E| \cdot (\lambda^2 \cdot W)^{2n})$), and exponential memory strategies are both necessary and sufficient for both players.

Proof : use the exponential reduction inspired from [CDRR15] and solve a generalized-Büchi $\cap$ co-Büchi game.

Reduction

Proposition

Each n-weighted game (G, Ω) *with* $\Omega = \cap_{m=1}^{n} \Omega_m$ *such that for all m,*

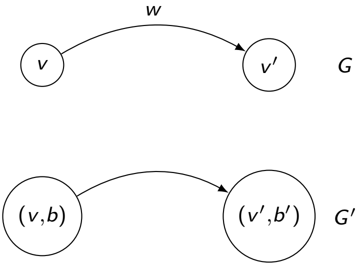
$$\Omega_m \in \{\text{WMP}, \text{Inf}, \text{Sup}, \text{LimInf}, \text{LimSup}\}$$

can be exponentially reduced to a game (G', Ω') *, and* $\Omega' = \cap_{m=1}^{n} \Omega'_m$ *such that for all m,*

$$\Omega'_m \in \{\text{Buchi}, \text{CoBuchi}\}.$$

Inf(0) case

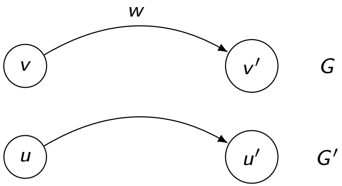
Idea: add a bit.
Bit equals -1 if a negative weight has been seen so far, equals 0 otherwise.



$$b' = \begin{cases} -1 & \text{if } b = -1 \text{ or } w < 0 \\ 0 & \text{otherwise} \end{cases}, \ \Omega' = \text{CoBuchi}(U) \text{ with } U = \{(v,b) \mid b = 0\}.$$

WMP(λ, 0) case

Idea: keep the current sum and the number of steps. Restart counters when window is first-closed.



From $u = (v, s, l)$, we have $(u, u') \in E'$ iff $(v, v') \in E$ and

$$u' = \begin{cases} (v', s + w(e), l + 1) & \text{if } s + w(e) < 0 \text{ and } l < \lambda - 1 \\ \beta & \text{if } s + w(e) < 0 \text{ and } l = \lambda - 1 \\ (v', 0, 0) & \text{if } s + w(e) \geq 0 \end{cases}.$$

$\Omega' = \text{CoBuchi}(V' \setminus \{\beta\})$ where β is an absorbing vertex informing of a bad window.

Even more

Theorem

Let (G, Ω) be an n-weighted game such that $\Omega = \cup_{k=1}^{d} \cap_{m=1}^{n} \Omega_{k,m}$ with $\Omega_{k,m} \in \{\text{WMP}, \text{Inf}, \text{Sup}, \text{LimInf}, \text{LimSup}\}$ for all k, m. Then, the threshold problem is EXPTIME-complete (with an algorithm in time $O((nd)^{d+2} \cdot |V|^{d+1} \cdot |E| \cdot (\lambda^2 \cdot W)^{nd(d+2)})$), and exponential memory strategies are both sufficient and necessary for both players.

Proof : Use the same reduction as before and solve a Rabin game with d pairs.

Remark: Undecidable even with $\Omega_{k,m} \in \{\overline{\text{MP}}; \underline{\text{MP}}\}$, for all k,m.

Objectives	Complexity class	Player 1 memory	Player 2 memory
Intersection of WMP, Inf, Sup, LimInf, LimSup	EXPTIME-complete	exponential	
Intersection of Inf, Sup, LimInf, LimSup and refinements	PSPACE-complete		
	Next Table		
WMP $\cap\ \Omega$ with $\Omega \in \{$Inf, Sup, LimInf, LimSup$\}$	P-complete	exponential	

Theorem

Let (G, Ω) be an n-weighted game such that $\Omega = \cap_{m=1}^{n} \Omega_m$ with $\Omega_m \in \{\text{Inf}, \text{Sup}, \text{LimInf}, \text{LimSup}\}$ for all m. Then the threshold problem is PSPACE-complete (with an algorithm in time $O(2^n \cdot (|V| + |E|))$) and exponential memory strategies are both necessary and sufficient for both players.

Proof: Use a polynomial reduction to obtain a game (G', Ω') with

$$\Omega' = \text{GenReach}(U_1, \dots, U_{j-1}) \cap \text{GenBuchi}(U_j, \dots, U_{i-1}) \cap \text{CoBuchi}(U_i).$$

Solve the generalized-Büchi ∩ co-Büchi game and then the generalized-reachability game.

Proposition

Each n-weighted game (G, Ω) *with* $G = (V_1, V_2, E, w)$, *and* $\Omega = \cap_{m=1}^{n} \Omega_m$ *such that for all m,*

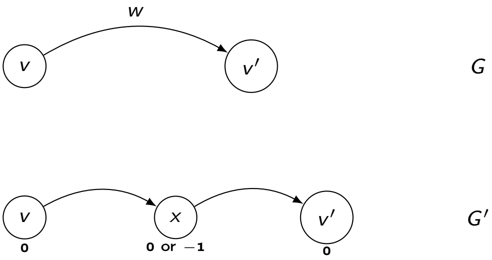
$$\Omega_m \in \{\mathsf{WMP}, \mathsf{Inf}, \mathsf{Sup}, \mathsf{LimInf}, \mathsf{LimSup}\}$$

can be polynomially reduced to a game (G', Ω') *with* $|V| + |E|$ *vertices and* $2 \cdot |E|$ *edges, and* $\Omega' = \cap_{m=1}^{n} \Omega'_m$ *such that for all m,*

$$\Omega'_m \in \{\mathsf{WMP}, \mathsf{Safe}, \mathsf{Reach}, \mathsf{CoBuchi}, \mathsf{Buchi}\}.$$

Case Inf(0)

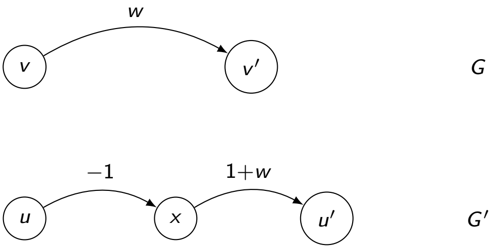
Idea: split the edges and color vertices.
Weight $< 0 \longleftrightarrow$ intermediate vertex decorated with -1
(0 otherwise)
Original vertices are decorated with 0.



$\Omega' = Safe(U)$ where U is the set of vertices decorated by 0

Case WMP(λ, 0)

Idea: split the weight w in -1 and $w+1$



$\Omega' = \text{WMP}(2\lambda, 0)$

Corollary

Inf	Sup	LimInf	LimSup	Complexity	player 1 memory	player 2 memory
any	any	any	any	PSPACE-c	finite-memory	finite-memory
any	≤ 1	any	any	P-complete	finite-memory	memoryless
any	0	any	≤ 1	P-complete	memoryless	memoryless
any	1	0	0	P-complete	memoryless	memoryless

Corollary

Inf	Sup	LimInf	LimSup	Complexity	player 1 memory	player 2 memory
any	any	any	any	PSPACE-c	finite-memory	finite-memory
any	≤ 1	any	any	P-complete	finite-memory	memoryless
any	0	any	≤ 1	P-complete	memoryless	memoryless
any	1	0	0	P-complete	memoryless	memoryless

We avoid having two WMP objectives (EXPTIME-hardness) and several Sup objectives (PSPACE-hardness).

Objectives	Complexity class	Player 1 memory	Player 2 memory
Intersection of WMP, Inf, Sup, LimInf, LimSup	EXPTIME-complete	exponential	
Intersection of Inf, Sup, LimInf, LimSup	PSPACE-complete	exponential	
and refinements	Previous Table		
WMP $\cap \Omega$ with $\Omega \in \{$Inf, Sup, LimInf, LimSup$\}$	P-complete	exponential	

Theorem

Let $G = (V_1, V_2, E, w)$ be a weighted game structure, and (G, Ω) be a game with objective $\Omega = \Omega_1 \cap \Gamma$ for player 1 such that $\Omega_1 = \text{WMP}$ and

- *either $\Gamma = \Omega_2 \in \{\text{Inf}, \text{Sup}, \text{LimInf}, \text{LimSup}\}$,*
- *or $\Gamma = \cap_{m=2}^{n} \Omega_m$ such that $\forall m$, $\Omega_m = \text{Inf}$ (resp. $\Omega_m = \text{LimInf}$, $\Omega_m = \text{LimSup}$).*

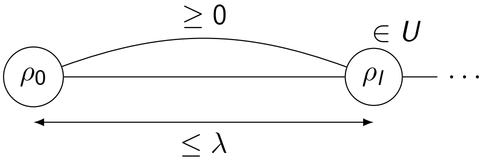
Then the threshold problem is P-complete. In general, both players require finite-memory and exponential memory is sufficient for both players.

Reduction: we study games $(G', \Omega'_1 \cap \Omega'_2)$ with $\Omega'_1 = \text{WMP}$ and $\Omega'_2 \in \{\text{Safe}, \text{Reach}, \text{Buchi}, \text{CoBuchi}, \text{GenBuchi}\}$.

- WMP $\cap$ Safe : easy
  Solve the Safe game and then the WMP game
- WMP $\cap\ \Omega_2 \in \{\text{Reach}, \text{Buchi}, \text{CoBuchi}, \text{GenBuchi}\}$: much more involved
  Need genuine new tools to deal with windows and the qualitative objectives.

First tool:

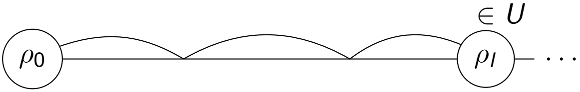
$\text{ICWEnd}^{\lambda}(U)$:



$\text{GDEnd}^{\lambda}(U)$

Second tool: Generalization of the p-attractor of a set U while dealing with windows.

$\text{GDEnd}^{\lambda}(U)$:



Require: 1-weighted game structure $G = (V_1, V_2, E, w)$, subset $U \subseteq V$, window size $\lambda \in \mathbb{N} \setminus \{0\}$

Ensure: $\text{Win}_1^{\text{GDEnd}^{\lambda}(U)}(G)$

```
k ← 0
X_0 ← U
repeat
    X_{k+1} ← X_k ∪ ICWEnd(G, X_k, λ)
    k ← k + 1
until X_k = X_{k−1}
return X_k
```

- WMP $\cap$ Reach:
  Use algorithm $\text{GDEnd}^{\lambda}(U')$ on a modified graph.
  U' is the set of vertices that denote that we have visited U and that are winning for the WMP objective.
- WMP $\cap$ Buchi:
  Roughly works as follows: compute the winning region for WMP $\cap$ Reach and repeat in the subgame.
- WMP $\cap$ GenBuchi
  Use a classical reduction from generalized-Büchi games to Büchi games.

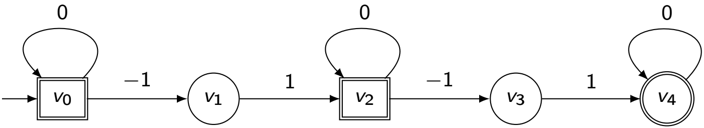
- WMP $\cap$ CoBuchi

Much more involved: need two nested fixpoints.
Ideas : vertices in $\text{Win}_1^{\text{WMP}\cap\text{Safe}}(G)$ are winning. This is also the case for vertices from which:

- player 1 can inductively-close the window in a vertex of $\text{Win}_1^{\text{WMP}\cap\text{Safe}}(G)$ or
- while staying in U, he can inductively-close the window in a vertex previously computed for the same properties.

Example

$\lambda = 2,\ U = \{v_0, v_2, v_4\},\ \Omega = \mathrm{WMP}(2,0) \cap \mathrm{CoBuchi}(U)$



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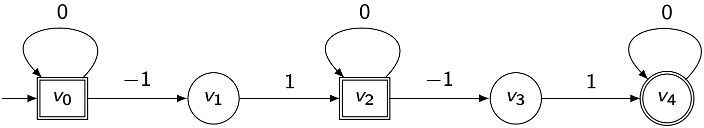

$X_0 = \emptyset, Z_{0,0} = V, Z_{0,1} = \{v_4\} = Z_{0,2}$

Example

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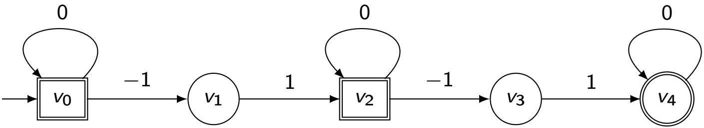

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$X_1 = \{v_4\}, Z_{1,0} = V, Z_{1,1} = \{v_2, v_3, v_4\} = Z_{1,2}$

Example

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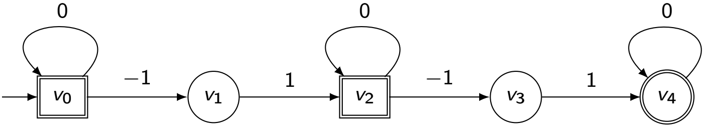

$X_0 = \emptyset, Z_{0,0} = V, Z_{0,1} = \{v_4\} = Z_{0,2}$
$X_1 = \{v_4\}, Z_{1,0} = V, Z_{1,1} = \{v_2, v_3, v_4\} = Z_{1,2}$
$X_2 = \{v_2, v_3, v_4\}, Z_{2,0} = V = Z_{2,1}$

Dimension n fixed (and d for first row).

Objectives	Complexity class	Parameterized complexity
Boolean combination (DNF) of WMP, Inf, Sup, LimInf, LimSup	EXPTIME-complete	
Intersection of WMP, Inf, Sup, LimInf, LimSup		
Intersection of Inf, Sup, LimInf, LimSup	PSPACE-complete	P-complete

Proof:
(Rows 1-2): Games with two WMP objectives are already EXPTIME-hard.
(Row 3): Generalized-reachability games are polynomial when the number of sets to visit is fixed.

Overview

Objectives	Complexity class	Player 1 memory	Player 2 memory
Boolean combination of $\underline{MP}$, $\overline{MP}$ [Vel15]	Undecidable	infinite	infinite
Intersection of WMP, Inf, Sup, LimInf, LimSup	EXPTIME-complete	exponential	
Intersection of WMP [CDRR15]			
Intersection of Inf, Sup, LimInf, LimSup and refinements	PSPACE-complete		
	Table of "PSPACE case"		
Intersection of $\underline{MP}$ [VCD+15]	coNP-complete	infinite	memoryless
Intersection of $\overline{MP}$ [VCD+15]	NP $\cap$ coNP		
Onedimensional MP [ZP96, BCD+11]		memoryless	
Onedimensional WMP [CDRR15]	P-complete	exponential	
WMP $\cap\ \Omega$ with $\Omega \in \{$Inf, Sup, LimInf, LimSup$\}$			
Onedimensional Inf, Sup, LimInf, LimSup [GTW02]		memoryless	

Table: Overview

Future Work

- General results for ω-regular objectives with WMP,
- Mix MP and Energy objectives.

- Value of WMP,
- Pareto curves.

Lubos Brim, Jakub Chaloupka, Laurent Doyen, Raffaella Gentilini, and Jean-François Raskin.
Faster algorithms for mean-payoff games.
Formal Methods in System Design, 38(2):97–118, 2011.

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