Simple strategies for Banach–Mazur games and sets of probability 1

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A B S T R A C T

In 2006, Varacca and Völzer proved that on finite graphs, \( \omega \)-regular large sets coincide with \( \omega \)-regular sets of probability 1, by using the existence of positional strategies in the related Banach–Mazur games. Motivated by this result, we try to understand relations between sets of probability 1 and various notions of simple strategies (including those introduced in a recent paper of Grädel and Leßenich). Then, we introduce a generalisation of the classical Banach–Mazur game and in particular, a probabilistic version whose goal is to characterise sets of probability 1 (as classical Banach–Mazur games characterise large sets). We obtain a determinacy result for these games, when the winning set is a countable intersection of open sets.

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1. Introduction

Systems (automatically) controlled by computer programs abound in our everyday life. Clearly enough, it is of a capital importance to know whether the programs governing these systems are correct. Over the last thirty years, formal methods for verifying computerised systems have been developed for validating the adequacy of the systems against their requirements. Model checking is one such approach: it consists first in modelling the system under study (for instance by an automaton), and then in applying algorithms for comparing the behaviours of that model against a specification (modelled for instance by a logical formula). Model checking has now reached maturity, through the development of efficient symbolic techniques, state-of-the-art tool support, and numerous successful applications to various areas.

As argued in [12]: 'Sometimes, a model of a concurrent or reactive system does not satisfy a desired linear-time temporal specification but the runs violating the specification seem to be artificial and rare'. As a naive example of this phenomenon, consider a coin flipped an infinite number of times. Classical verification will assure that the property stating “one day, we will observe at least one head” is false, since there exists a unique execution of the system violating the property. In some situations, for instance when modelling non-critical systems, one could prefer to know whether the system is fairly correct. Roughly speaking, a system is fairly correct against a property if the set of executions of the system violating the property is "very small"; or equivalently if the set of executions of the system satisfying the property is "very big". A first natural notion of a fairly correct system is related to probability: almost-sure correctness. A system is almost-surely correct against a property if the set of executions of the system satisfying the property has probability 1. Another interesting notion of fairly correct

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system is related to topology: large correctness. A system is largely correct against a property if the set of executions of the system satisfying the property is large (in the topological sense). There exists a nice characterisation of large sets by means of the Banach–Mazur games. In [10], it has been shown that a set \( W \) is large if and only if Player 0 has a winning strategy in the related Banach–Mazur game.

Although, the two notions of fairly correct systems do not coincide in general, in [12], the authors proved (amongst other results) the following result: when considering \( \omega \)-regular properties on finite systems, the almost-sure correctness and the large correctness coincide, for bounded Borel measures. Motivated by this very nice result, we intend to extend it to a larger class of specifications. The key ingredient to prove the previously mentioned result of [12] is that when considering \( \omega \)-regular properties, positional strategies are sufficient in order to win the related Banach–Mazur game [2]. For this reason, we investigate simple strategies in Banach–Mazur games, inspired by the recent work [5] where infinite graphs are studied.

Our contributions. In this paper, we first compare various notions of simple strategies on finite graphs (including bounded and move-counting strategies), and their relations with the sets of probability 1. Given a set \( W \), the existence of a bounded (resp. move-counting) winning strategy in the related Banach–Mazur game implies that \( W \) is a set of probability 1. However there exist sets \( W \) of probability 1 for which there is no bounded and no move-counting winning strategy in the related Banach–Mazur game. Therefore, we introduce a generalisation of the classical Banach–Mazur game and in particular, a probabilistic version whose goal is to characterise sets of probability 1 (as classical Banach–Mazur games characterise large sets). We obtain the desired characterisation in the case of countable intersections of open sets. This is the main contribution of the paper. As a byproduct of the latter, we get a determinacy result for our probabilistic version of the Banach–Mazur game for countable intersections of open sets. We finally extend this probabilistic version to Banach–Mazur games on topological spaces.

This paper is an extended version of the paper “Simple strategies for Banach–Mazur games and fairly correct system” published in the proceedings of GandALF 2013 [4]. In this version, we have added a comparison of move/length-counting strategies by showing that one is a uniform version of the other. We have investigated the minimal necessary memory to keep the full power of winning strategies and we have extended our notion of generalised Banach–Mazur games (originally only defined on finite graphs) to topological spaces.

2. Banach–Mazur games on finite graphs

2.1. General background

Let \((X, T)\) be a topological space. We first recall that the closure of a subset \( W \subset X \), denoted by \( \overline{W} \) is the smallest closed set containing \( W \) and that the interior of \( W \), denoted by \( \text{int}(W) \), is the biggest open set contained in \( W \). A notion of topological “bigness” is then given by large sets. A subset \( W \subset X \) is said to be nowhere dense if the closure of \( W \) has empty interior. A subset \( W \subset X \) is said to be meagre if it can be expressed as the union of countably many nowhere dense sets and a subset \( W \subset X \) is said to be large if \( W^c \) is meagre. In particular, we remark that a countable intersection of large sets is still large and that if \( W \subset X \) is large, then any set \( Y \supset W \) is large.

If \( G = (V, E) \) is a finite directed graph and \( v_0 \in V \), then the space of infinite paths in \( G \) from \( v_0 \), denoted \( \text{Paths}(G, v_0) \), can be endowed with the complete metric

\[
d((\sigma_n)_{n \geq 0}, (\rho_n)_{n \geq 0}) = 2^{-k} \quad \text{where} \quad k = \min\{n \geq 0 : \sigma_n \neq \rho_n\}
\]

with the conventions that \( \min\emptyset = \infty \) and \( 2^{-\infty} = 0 \). In other words, the open sets in \( \text{Paths}(G, v_0) \) endowed with this metric are the unions of cylinders, where a cylinder is a set of the form \( \{\rho \in \text{Paths}(G, v_0) \mid \pi \text{ is a prefix of } \rho \} \) for some finite path \( \pi \) in \( G \) from \( v_0 \).

We can therefore study the large subsets of the metric space \( \text{Paths}(G, v_0), d \). Banach–Mazur games allow us to characterise large subsets of this metric space through the existence of winning strategies.

Definition 2.1. A Banach–Mazur game \( G \) on a finite graph is a triplet \( (G, v_0, W) \) where \( G = (V, E) \) is a finite directed graph where every vertex has a successor, \( v_0 \in V \) is the initial state, \( W \) is a subset of the infinite paths in \( G \) starting in \( v_0 \).

A Banach–Mazur game \( G = (G, v_0, W) \) on a finite graph is a two-player game where \( P.0 \) and \( P.1 \) alternate in choosing a finite path as follows: \( P.1 \) begins with choosing a finite \(^2\) path \( \pi_1 \) starting in \( v_0 \); \( P.0 \) then prolongs \( \pi_1 \) by choosing another finite path \( \pi_2 \) and so on. A play of \( G \) is thus an infinite path in \( G \) and we say that \( P.0 \) wins if this path belongs to \( W \), while \( P.1 \) wins if this path does not belong to \( W \). The set \( W \) is called the winning condition. It is important to remark that, in general, in the literature, \( P.0 \) moves first in Banach–Mazur games but in this paper, we always assume that \( P.1 \) moves first in order to bring out the notion of large set (rather than meagre set). The main result about Banach–Mazur games can then be stated as follows:

\(^2\) In this paper, we always assume that a finite path is non-empty.
Theorem 2.2. (See [10].) Let $G = (G, v_0, W)$ be a Banach–Mazur game on a finite graph. Pl. 0 has a winning strategy for $G$ if and only if $W$ is large.

2.2. Simple strategies in Banach–Mazur games

Given a finite path $\pi = v_1 \cdots v_k$, we denote by first($\pi$) $= v_1$ and by last($\pi$) $= v_k$ the first and last vertices of $\pi$. For all nodes $v \in V$ we then denote by FinPaths($G, v$) the set of finite paths $\pi$ such that $(v, \text{first}(\pi)) \in E$ and we denote by FinPlays($G, v$) the set of finite plays from $v$, i.e. the finite sequences of finite paths $(\pi_1, \ldots, \pi_k)$ such that $\pi_1 \in$ FinPaths($G, v$) and for all $1 \leq i \leq k$, $\pi_i \in$ FinPaths($G, \text{last}(\pi_{i-1})$).

In a Banach–Mazur game $(G, v_0, W)$ on a finite graph, a strategy for Pl. 0 is given by a function $f$ defined on FinPlays($G, v_0$), such that for all $(\pi_1, \ldots, \pi_k) \in$ FinPlays($G, v_0$), we have $f(\pi_1, \ldots, \pi_k) \in$ FinPlays($G, \text{last}(\pi_k)$).

However, we can imagine some restrictions on the strategies of Pl. 0. Given a (possibly infinite) set $S$ and a mapping $m :$ FinPlays($G, v_0$) $\rightarrow S$, we say that a strategy $f$ is $m$-restricted if for all $(\pi_1, \ldots, \pi_k)$, $(\pi_1', \ldots, \pi'_k)$ such that $\pi(\pi_1, \ldots, \pi_k) = m(\pi_1', \ldots, \pi'_k)$ and last($\pi_k$) = last($\pi'_k$), we have $f(\pi_1, \ldots, \pi_k) = f(\pi_1', \ldots, \pi'_k)$. In this case we abuse the notation by defining $f$ on $V \times S$ such that $f(\pi, s)$ is equal to $f(\pi_1, \ldots, \pi_k)$ for all $(\pi_1, \ldots, \pi_k)$ such that last($\pi_k$) $= v$ and $m(\pi_1, \ldots, \pi_k) = s$.

- A strategy $f$ is move-blind\(^3\) if it is $m_{\text{mb}}$-restricted with $m_{\text{mb}} :$ FinPlays($G, v_0$) $\rightarrow$ FinPaths($G, v_0$) defined by $m_{\text{mb}}(\pi_1, \ldots, \pi_k) = \pi_1 \cdots \pi_k$. In this context we simply write $f(\pi)$ for $f(\pi, \pi)$ (since $v$ will always be equal to last($\pi$)).
- A strategy $f$ is move-counting if it is $m_{\text{mc}}$-restricted with $m_{\text{mc}} :$ FinPlays($G, v_0$) $\rightarrow$ $\mathbb{N}$ defined by $m_{\text{mc}}(\pi_1, \ldots, \pi_k) = k$.
- A strategy $f$ is length-counting if it is $m_{\text{lc}}$-restricted with $m_{\text{lc}} :$ FinPlays($G, v_0$) $\rightarrow$ $\mathbb{N}$ defined by $m_{\text{lc}}(\pi_1, \ldots, \pi_k) = |\pi_1 \cdots \pi_k|$.
- A strategy $f$ is positional if it is $m_{\text{pos}}$-restricted with $m_{\text{pos}} :$ FinPlays($G, v_0$) $\rightarrow$ $\{\bot\}$ defined by $m_{\text{pos}}(\pi_1, \ldots, \pi_k) = \bot$. In this case we simply write $f(\pi)$ for $f(\pi, \bot)$.

Given a set $S$ and a mapping $m :$ FinPlays($G, v_0$) $\rightarrow S$, we extend the notion of move-blindness to mappings by saying that $m$ is move-blind if for all $(\pi_1, \ldots, \pi_k)$ and $(\pi_1', \ldots, \pi'_k)$ such that $\pi_1 \cdots \pi_k = \pi_1' \cdots \pi'_k$, we have $m(\pi_1, \ldots, \pi_k) = m(\pi_1', \ldots, \pi'_k)$. Note that $m_{\text{mb}}, m_{\text{mc}}$ and $m_{\text{pos}}$ are move-blind while $m_{\text{lc}}$ is not, and that if a strategy $f$ is $m$-restricted with $m$ being a move-blind mapping then $f$ is move-blind.

Given a set $S$ and a mapping $m :$ FinPlays($G, v_0$) $\rightarrow S$, we say that $m$ is a memory machine if it is move-blind and for all $\pi, \pi'$, such that $m(\pi) = m(\pi')$, we have $m(\pi \nu) = m(\pi' \nu)$ for all $\nu$. In that case, if $s = m(\pi) = m(\pi')$, we write $m(s, \nu) = m(\pi \nu) = m(\pi' \nu)$ and for all finite path $\pi''$ we write $m(s, \pi'') = m(\pi \pi'') = m(\pi' \pi'')$. Furthermore we say that $m$ is finite-memory if $S$ is finite. Note that $m_{\text{mb}}, m_{\text{mc}}$ and $m_{\text{pos}}$ are memory machines, and $m_{\text{pos}}$ is finite-memory.

- A strategy $f$ is finite-memory if it is $m$-restricted for some finite-memory machine $m$ (see [6] for more details on finite-memory strategies).

Finally we can also restrict the output of the strategies as follows.

- A strategy $f$ is said to be $b$-bounded if for all $(\pi_1, \ldots, \pi_k) \in$ FinPlays($G, v_0$), $f(\pi_1, \ldots, \pi_k)$ has length less than $b$ and a strategy is said to be bounded if there is $b \geq 1$ such that $f$ is $b$-bounded.

The notions of positional and finite memory strategies are classical, bounded strategies are present in [12] and move-counting and length-counting strategies have been introduced in [5].

2.3. Comparing the classes of strategies

One of the main properties of Banach–Mazur games is that if Pl. 0 has a winning strategy, then Pl. 0 has a move-blind winning strategy [6]. Therefore in the following, when no other restriction is stated, we always assume that the strategies we consider are move-blind.

Remark that, by definition, the existence of a positional winning strategy implies the existence of a finite-memory/move-counting/length-counting winning strategies. Moreover, since $G$ is a finite graph, a positional strategy is always bounded.

To establish some equivalences between classes of strategies, we introduce a contraction relation between restricting mappings, as follows.

Let $m :$ FinPlays($G, v_0$) $\rightarrow S$ and $m' :$ FinPlays($G, v_0$) $\rightarrow S'$ be two mappings. We say that $m'$ is a contraction of $m$ if for all $s' \in S'$, the set $\{s : \exists \pi m'(\pi) = s' \land m(\pi) = s\}$ is finite. Intuitively, after reading a path $\pi$, without having to compute

\(^3\) Also called decomposition invariant in [6].
Lemma 2.3. Let \((G, v_0, W)\) be a Banach–Mazur game, \(m : \text{FinPlays}(G, v_0) \rightarrow S\) and \(m' : \text{FinPlays}(G, v_0) \rightarrow S'\) be two mappings such that \(m\) is a contraction of \(m\) and \(m\) is a memory machine. If \(Pl. 0\) has an \(m\)-restricted winning strategy, then \(Pl. 0\) has an \(m'\)-restricted winning strategy.

Proof. Let \(f\) be an \(m\)-restricted winning strategy for \(Pl. 0\). For all \(s' \in S\) we can consider an enumeration \(s_1, \ldots, s_k\) of the set \(\{s | \exists \pi \ m' (\pi) = s' \land m(\pi) = s\}\). We then define an \(m'\)-restricted strategy \(h\) as follows:

\[
h(v, s') = \begin{cases} 
\pi_2 \\
\pi_1
\end{cases}
\]

\[
\pi_1 = \begin{cases} 
n \text{last}(\pi_1), m(s_2, \pi_1) \cdots \text{last}(\pi_{k-1}), m(s_k, \pi_{k-1}) \end{cases}
\]

If \(\rho\) is a play consistent with \(h\), then \(\rho\) is a play where the strategy \(f\) is applied infinitely often. Thus such a play \(\rho\) can be seen as a play \(\sigma_1 \tau_1 \sigma_2 \tau_2 \cdots\) where the \(\tau_i\)'s (resp. the \(\pi_i\)'s) are the moves of \(Pl. 0\) (resp. \(Pl. 1\)) and where \(f(\sigma_1 \tau_1 \cdots \sigma_j) = \tau_j\). Each play consistent with \(h\) can thus be seen as a play consistent with \(f\), and we deduce that the strategy \(h\) is an \(m'\)-restricted winning strategy. \(\square\)

In [6], this technique is used to prove that the existence of a finite-memory winning strategy implies the existence of a positional winning strategy.

Proposition 2.4. (See [6]) Let \(G = (G, v_0, W)\) be a Banach–Mazur game. \(Pl. 0\) has a finite-memory winning strategy if and only if \(Pl. 0\) has a positional winning strategy.

Proof. Simply notice that a finite-memory machine \(m, m_{pos}\) is a contraction of \(m\). \(\square\)

We can also show, using the same lemma, that the existence of a winning strategy implies the existence of a length-counting winning strategy.

Proposition 2.5. Let \(G = (G, v_0, W)\) be a Banach–Mazur game on a finite graph. \(Pl. 0\) has a length-counting winning strategy if and only if \(Pl. 0\) has a winning strategy.

Proof. Simply notice that \(m_{kb}\) is a contraction of \(m_{nb}\). Indeed, since \(G\) is a finite graph, for all \(\ell\) there are only finitely many paths of length \(\ell\). \(\square\)

On the other hand, the notions of move-counting winning strategies and bounded winning strategies are incomparable.

Example 2.6 (Set with a move-counting winning strategy and without a bounded winning strategy). We consider the complete graph \(G_{0,1}\) on \([0, 1]\). Let \(W\) be the set of sequences \((\sigma_n)_{n \geq 1}\) in \([0, 1]^\omega\) with \(\sigma_1 = 0\) such that \((\sigma_n)_{n \geq 1}\) contains a finite sequence of \(1\) strictly longer than the initial finite sequence of \(0\). In other words, \((\sigma_n)_{n \geq 1} \in W\) if \(\sigma_1 = 0\) and if there exist \(j \geq 1\) and \(k \geq 1\) such that \(\sigma_j = 1\) and \(\sigma_{k+1} = \cdots = \sigma_{k+j} = 1\). Let \(G = (G_{0,1}, 0, W)\). The strategy \(f(\cdot, n) = 1^n\) is a move-counting winning strategy for \(Pl. 0\) for the game \(G\). On the other hand, there does not exist a bounded winning strategy for \(Pl. 0\) for the game \(G\). Indeed, if \(f\) is a \(b\)-bounded strategy of \(Pl. 0\), then \(Pl. 1\) can start by playing \(0^b\) and then, always play 0.

Example 2.7 (Set with a bounded winning strategy and without a move-counting winning strategy). We consider the complete graph \(G_{0,1}\) on \([0, 1]\). Let \((\pi_n)_{n \geq 0}\) be an enumeration of \(\text{FinPaths}(G)\) with \(\pi_0 = 0\). We let \(W\) be the set of sequences in \([0, 1]^\omega\) starting by 0 except the sequence \(\rho = \pi_0 \pi_1 \pi_2 \ldots\). Let \(G = (G_{0,1}, 0, W)\). It is obvious that \(Pl. 0\) has a 1-bounded winning strategy for \(G\) but we can also prove that \(Pl. 0\) has no move-counting winning strategy. Indeed, if \(h\) is a move-counting strategy of \(Pl. 0\), then \(Pl. 1\) can start by playing a prefix \(\pi\) of \(\rho\) so that \(\pi h(\text{last}(\pi)), 1)\) is a prefix of \(\rho\). Afterwards, \(Pl. 1\) can play \(\pi'\) such that \(\pi h(\text{last}(\pi), 1) \pi' h(\text{last}(\pi'), 2)\) is a prefix of \(\rho\) and so on.

We remark that the sets \(W\) considered in these examples are open sets, i.e. sets on a low level of the Borel hierarchy. Moreover, by Proposition 2.5, there also exist length-counting winning strategies for these two examples. The relations between the simple strategies are thus completely characterised and are summarised in Fig. 1.
2.4. Alternative characterisation of move/length-counting strategies

In the following we give some alternative characterisations of the move-counting and length-counting strategies. This allows us to express move-counting strategies as move-blind strategies, and to link the two classes of strategies as one being a uniform version of the other.

To this end, we introduce a class of strategies, called the clock-strategies defined as follows. A clock strategy is a pair $\langle (f_i)_{i\in\mathbb{N}}, c \rangle$ defined by $\langle (f_i)_{i\in\mathbb{N}}, c \rangle(\pi) = f_{c(|\pi|)}(\text{last}(\pi))$, where $(f_i)_{i\in\mathbb{N}}$ is a sequence of positional strategies and $c : \mathbb{N} \to \mathbb{N}$ is an unbounded non-decreasing mapping called a clock. Intuitively when Pl. 0 is called on the finite path $\pi$, the clock $c$ takes as input the length of $\pi$ and outputs the index $c(|\pi|) = n$ of a positional strategy $f_n$ that is called on the last vertex of $\pi$. We can see a clock-strategy $\langle (f_i)_{i\in\mathbb{N}}, c \rangle$ as an $m_c$-restricted strategy $f_{\langle (f_i)_{i\in\mathbb{N}}, c \rangle}$ given by $f_{\langle (f_i)_{i\in\mathbb{N}}, c \rangle}(v, n) = f_n(v)$, where for each clock $c$ we denote by $m_c$ the mapping defined by $m_c(\pi) = c(|\pi|)$.

The reason why we assume that the clocks are unbounded is that if the clocks are bounded, we have the same power as positional strategies, i.e. there exists a winning clock-strategy $\langle (f_i)_{i\in\mathbb{N}}, c \rangle$ for some bounded clock $c$ if and only if there exists a positional winning strategy. Indeed if $c$ is bounded, $m_c$ can be simulated by a finite-memory machine and we conclude by Proposition 2.4.

Using clock-strategies we give a new characterisation of move/length-counting winning strategies, saying that there exists a move-counting winning strategy if and only if there exists a family of strategies that is winning paired with any clock, and there exists a length-counting winning strategy if and only if for any clock, there exists a clock-strategy equipped with this clock that is winning. These results are summarised in the following theorem.

**Theorem 2.8.** Let $(G, v_0, W)$ be Banach–Mazur game. The following equations hold.

There exists a length-counting winning strategy $\iff \forall c \in \text{Clocks} \exists (f_i)_{i\in\mathbb{N}} \langle (f_i)_{i\in\mathbb{N}}, c \rangle$ wins

There exists a move-counting winning strategy $\iff \exists (f_i)_{i\in\mathbb{N}} \forall c \in \text{Clocks} \langle (f_i)_{i\in\mathbb{N}}, c \rangle$ wins

where the $f$’s are positional strategies.

**Proof.** We start by proving the first equivalence. To prove the left to right implication, let us observe that for any clock $c \in \text{Clocks}$, $m_c$ is a contraction of $m_{lc}$. Indeed, as $c$ is non-decreasing and unbounded, we know that for all $n$, there are only finitely many $\ell$ such that $c(\ell) = n$. Then we can apply Lemma 2.3 since $m_{lc}$ is a memory machine. To prove the right to left implication, it suffices to consider the clock $c(n) = n$.

Let us now prove the second equivalence. Assume that there exists a move-counting winning strategy $f$. For all vertices $v$, we fix a vertex $\text{succ}(v)$ such that $(v, \text{succ}(v)) \in E$ and for all finite paths $\pi$ we define $\text{succ}(\pi) = \text{succ}(\text{last}(\pi))$. For all $n$ we can then define a positional strategy $f_n$ as follows:

$$f_n(v) = f(v, 1)v_1f(v_1, 2)v_2f(v_2, 3)v_3\cdots f(v_n, n)$$

where $v_1 = \text{succ}(f(v, 1))$ and for all $i > 1$, $v_i = \text{succ}(f(v_{i-1}, i))$. One can easily see that for any clock $c$, if a play is consistent with $f_{\langle (f_i)_{i\in\mathbb{N}}, c \rangle}$ then it is consistent with $f$, which concludes the proof of the left to right implication.
Assume now that there exists \((f_i)_{i \in \mathbb{N}}\), a sequence of positional strategies, such that for all \(c \in \text{Clocks}\), the clock-strategy defined by \(\langle (f_i)_{i \in \mathbb{N}}, c \rangle\) is winning for Pl. 0. We now define a move-counting strategy \(f\) as follows: \(f(v, i) = f_i(v)\). The strategy \(f\) is winning for Pl. 0. Indeed, given a play \(\rho\) consistent with \(f\), we know that \(\rho\) is of the form

\[
\pi_1 f_1(\text{last}(\pi_1)) \pi_2 f_2(\text{last}(\pi_2)) \pi_3 f_3(\text{last}(\pi_3)) \ldots
\]

and this play can be seen as play consistent with the clock-strategy \(\langle (f_i)_{i \in \mathbb{N}}, c \rangle\), where \(c\) is a clock satisfying \(c(\pi_1 f_1(\text{last}(\pi_1)) \ldots \pi_n) = n\). We conclude that \(\rho\) belongs to \(W\), as \(\langle (f_i)_{i \in \mathbb{N}}, c \rangle\) is winning by assumption. \(\Box\)

One way to interpret this result is that move-counting strategies are the weakest infinite memory strategies, i.e. basically all they require is that each positional strategy in the family is played at some point.

3. Link with the sets of probability 1

Let \(G = (V, E)\) be a finite directed graph. We can easily define a probability measure \(P\), on the set of infinite paths in \(G\), by giving a weight \(w_e > 0\) to each edge \(e \in E\) and by considering that for all \(v, v' \in V\), \(p_w(v, v') = 0\) if \((v, v') \notin E\) and \(p_w(v, v') = \sum_{v' \in E} w_{vv'}\) else, where \(p_w(v, v')\) denotes the probability of taking edge \((v, v')\) from state \(v\). Given \(v_1 \cdots v_n \in \text{FinPaths}(G, v_1)\), we recall that we denote by \(\text{Cyl}(v_1 \cdots v_n)\) the cylinder generated by \(v_1 \cdots v_n\) and defined as \(\text{Cyl}(v_1 \cdots v_n) = \{\rho \in \text{Paths}(G, v_1) \mid v_1 \cdots v_n \text{ is a prefix of } \rho\}\).

**Definition 3.1.** Let \(G = (V, E)\) be a finite directed graph and \(w = (w_e)_{e \in E}\) a family of positive weights. We define the probability measure \(P_w\) by the relation

\[
P_w(\text{Cyl}(v_1 \cdots v_n)) = p_w(v_1, v_2) \ldots p_w(v_{n-1}, v_n)
\]

and we say that such a probability measure is **reasonable**.

We are interested in characterising the sets \(W\) of probability 1 and their links with the different notions of simple winning strategies. We remark that, in general, Banach–Mazur games do not characterise sets of probability 1. In other words, the notions of large sets and sets of probability 1 do not coincide in general on finite graphs. Indeed, there exist some large sets of probability 0 (which, by taking the complement, is equivalent to saying that there exist some sets of probability 1 that are meagre). We present here an example of such sets.

**Example 3.2 (Large set of probability 0).** We consider the complete graph \(G_{0,1.2}\) on \([0, 1, 2]\) and the set \(W = \{(w_i w_e^R)_{i \geq 1} \in \text{Paths}(G_{0,1.2}, 2) : w_i \in [0, 1, 2]^*\}\), where for all finite word \(\sigma \in [0, 1, 2]^*\) given by \(\sigma = \sigma(1) \cdots \sigma(n)\) with \(\sigma(i) \in [0, 1, 2]\), we let \(\sigma^R = \sigma(n) \cdots \sigma(1)\). In other words, \(W\) is the set of runs \(\rho\) starting from 2 that we can divide into a consecutive sequence of finite words and their reverses. It is obvious that Pl. 0 has a winning strategy for the Banach–Mazur game \((G_{0,1.2}, 2, W)\) and thus that \(W\) is large. On the other hand, if \(P\) is the reasonable probability measure by the weights \(w_e = 1\) for all \(e \in E\), then we can verify that \(P(W) = 0\). Indeed, we have

\[
P(W) \leq \sum_{n=1}^{\infty} P(\{w_0 w_1^R, \ldots w^n_1 w_e^R \}_{i \geq 1} \in W : |w_0| = n)
\]

\[
= \sum_{n=1}^{\infty} P(\{w_0 w_1^R w \in \text{Paths}(G_{0,1.2}, 2) : |w_0| = n\}) \cdot P(W)
\]

\[
\leq \sum_{n=1}^{\infty} P(W) \frac{1}{2^n} = \frac{1}{2} P(W).
\]

For certain families of sets, we can however have an equivalence between the notion of large set and the notion of set of probability 1. It is the case for the family of sets \(W\) representing \(\omega\)-regular properties on finite graphs (see [12]). In order to prove this equivalence for \(\omega\)-regular sets, Varacca and Völzer have in fact used the fact that for these sets, the Banach–Mazur game is positionally determined [2] and that the existence of a positional winning strategy for Pl. 0 implies \(P(W) = 1\). This latter assertion follows from the fact that every positional strategy is bounded and that, by the Borel–Cantelli lemma, the set of plays consistent with a bounded strategy is a set of probability 1. Nevertheless, if \(W\) does not represent an \(\omega\)-regular property, it is possible that \(W\) is a large set of probability 1 and that there is no positional winning strategy for Pl. 0 and even no bounded or move-counting winning strategy.

**Example 3.3 (Large set of probability 1 without a positional/bounded/move-counting winning strategy).** We consider the complete graph \(G_{0,1}\) on \([0, 1]\) and the reasonable probability measure \(P\) given by \(w_e = 1\) for all \(e \in E\). Let \(a_n = \sum_{k=1}^n k\). We let
Let \( W = \{ (\sigma_k)_{k \geq 1} \in [0,1]^\omega : \sigma_1 = 0 \) and \( \sigma_{an} = 1 \) for some \( n > 1 \} \) and \( G = (G_0, 0, W) \). Since Pl 0 has a winning strategy for \( G \), we deduce that \( W \) is a large set. We can also compute that \( P(W) = 1 \) because if we denote by \( A_n \), \( n > 1 \), the set
\[
A_n := \{ (\sigma_k)_{k \geq 1} \in [0,1]^\omega : \sigma_{bn} = 1 \) and \( \sigma_{bn} = 0 \) for all \( m < n \},
\]
we have:
\[
W = \bigcup_{n > 1} A_n \quad \text{and} \quad P(A_n) = \frac{1}{2^{n-1}}.
\]
On the other hand, there does not exist any positional (resp. bounded) winning strategy \( f \) for Pl 0. Indeed, if \( f \) is a positional (resp. bounded) strategy for Pl 0 such that \( f(0) \) (resp. \( f(\pi) \) for all \( \pi \)) has length less than \( n \), then Pl 1 just has to start by playing \( a_n \) zeros so that Pl 1 does not reach the index \( a_{n+1} \) and afterwards to complete the sequence by a finite number of zeros to reach the next index \( a_k \), and so on. Moreover, there does not exist any move-counting winning strategy \( h \) for Pl 0 because Pl 1 can start by playing \( a_n \) zeros so that \( |h(0,1)| \leq n \) and because, at each step \( k \), Pl 1 can complete the sequence by a finite number of zeros to reach a new index \( a_n \) such that \( |h(0,k)| \leq n \).

On the other hand, we can show that the existence of a move-counting winning strategy for Pl 0 implies \( P(W) = 1 \). The key idea is to realise that giving a move-counting winning strategy \( h \), the strategy \( h(\cdot, n) \) is positional.

**Proposition 3.4.** Let \( G = (G, v_0, W) \) be a Banach–Mazur game on a finite graph and \( P \) a reasonable probability measure. If Pl 0 has a move-counting winning strategy for \( G \), then \( P(W) = 1 \).

**Proof.** Let \( h \) be a move-counting winning strategy of Pl 0. We denote by \( f_n \) the positional strategy \( h(\cdot, n) \). Each set
\[
M_n := \{ \rho \in \text{Paths}(G, v_0) : \rho \text{ is a play consistent with } f_n \}
\]
has probability 1 since \( f_n \) is a positional winning strategy for the Banach–Mazur game \((G, v_0, M_n)\). Moreover, if \( \rho \) is a play consistent with \( f_n \) for each \( n \geq 1 \), then \( \rho \) is a play consistent with \( h \). In other words, since \( h \) is a winning strategy, we get \( \bigcap_n M_n \subseteq W \). Therefore, as \( P(M_n) = 1 \) for all \( n \), we know that \( P(\bigcap_n M_n) = 1 \) and we conclude that \( P(W) = 1 \).

Let us notice that the converse of Proposition 3.4 is false in general. Indeed, Example 3.3 exhibits a large set \( W \) of probability 1 such that Pl 0 has no move-counting winning strategy. However, if \( W \) is a countable intersection of \( \omega \)-regular sets, then the existence of a winning strategy for Pl 0 implies the existence of a move-counting winning strategy for Pl 0.

**Proposition 3.5.** Let \( G = (G, v_0, W) \) be a Banach–Mazur game on a finite graph where \( W \) is a countable intersection of \( \omega \)-regular sets \( W_n \). Pl 0 has a winning strategy if and only if Pl 0 has a move-counting winning strategy.

**Proof.** Let \( W = \bigcap_{n \geq 1} W_n \) where \( W_n \) is an \( \omega \)-regular set and \( f \) a winning strategy of Pl 0 for \( G \). For all \( n \geq 1 \), the strategy \( f \) is a winning strategy for the Banach–Mazur game \((G, v_0, W_n)\). Thanks to [2], we know that for all \( n \geq 1 \), there exists a positional winning strategy \( f_n \) of Pl 0 for \((G, v_0, W_n)\).

Let \( \psi : \mathbb{N} \to \mathbb{N} \) be such that for all \( k \geq 1 \), \( \{ n \in \mathbb{N} : \psi(n) = k \} \) is an infinite 4 set. We consider the move-counting strategy \( h(\cdot, n) = f_{\psi(\cdot)}(\cdot) \). This strategy is winning because each play \( \rho \) consistent with \( h \) is a play consistent with \( f_n \) for all \( n \) and thus
\[
\{ \rho \in \text{Paths}(G, v_0) : \rho \text{ is a play consistent with } h \} \subseteq \bigcap_n \{ \rho \in \text{Paths}(G, v_0) : \rho \text{ is a play consistent with } f_n \} \Rightarrow \bigcap_n W_n = W.
\]

**Remark 3.6.** We cannot extend this result to countable unions of \( \omega \)-regular sets because the set of countable unions of \( \omega \)-regular sets contains the open sets and Example 2.7 exhibited a Banach–Mazur game where \( W \) is an open set and Pl 0 has a winning strategy but no move-counting winning strategy.

**Remark 3.7.** We also notice that if \( W \) is a countable intersection of \( \omega \)-regular sets, then \( W \) is large if and only if \( W \) is a set of probability 1. Indeed, the notions of large sets and sets of probability 1 are stable by countable intersection and we know that an \( \omega \)-regular set is large if and only if it is of probability 1 [12].

---

4 Such a map \( \psi \) exists because one could build a surjection \( \psi : \mathbb{N} \to \mathbb{N} \times \mathbb{N} \) and then let \( \phi = \psi_1 \) where \( \psi(n) = (\psi_1(n), \psi_2(n)) \).

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As a consequence of Remark 3.7, we have that if \( W \) is an \( \omega S \)-regular sets, as defined in [1], the set \( W \) is large if and only if \( W \) is a set of probability 1. Indeed, it is shown in [8,9] that \( \omega S \)-regular sets are countable intersections of \( \omega \)-regular sets. Nevertheless, the following example shows that, unlike the case of \( \omega \)-regular sets, positional strategies are not sufficient for \( \omega S \)-regular sets.

Example 3.8 (\( \omega S \)-regular set with a move-counting winning strategy and without a positional/bounded winning strategy). We consider the complete graph \( G_{0,1} \) on \([0,1]\) and the set \( W \) corresponding to the \( \omega S \)-regular expression \((0^*1)^{\omega^2}1)^{\omega} \), which corresponds to the language of words where the number of consecutive 0 is unbounded. The move-counting strategy which consists in playing \( n \) consecutive 0’s at the \( n \)th step is winning for Pl. 0. However, clearly enough Pl. 0 does not have a positional (or bounded) winning strategy for \( W \).

A natural question is: “Are there other families of sets for which we have an equivalence between the existence of winning strategies and the existence of bounded/move-counting strategies?” For instance, we can consider the family of sets recognised by a deterministic counter-automaton. In a nutshell, a deterministic counter-automaton is a finite state-machine reading an infinite word while incrementing or decrementing a counter. It contains some rules deciding what to do (incrementing or decrementing) depending on the current state, on the current letter, and on whether the value of the counter is zero or not. The acceptance condition of such an automaton is given by a finite set of configurations of the automaton, and a word is accepted if infinitely often some of these configurations are reached. Unfortunately, we can easily exhibit a counter-example for this family.

Example 3.9 (Set recognised by a deterministic counter-automaton with a winning strategy but no bounded winning strategy or move-counting winning strategy). We consider the complete graph \( G_{0,1} \) on \([0,1]\) and the set \( W \) of words of \([0,1]^{\omega} \) recognised by an automaton that increments the counter when it reads 1, decrements it when it reads 0 and the value of the counter is greater that 0, does nothing when it reads 0 and the value of the counter is 0, and accepts a word if infinitely often the value of the counter is 0.

First, we remark that Pl. 0 has a winning strategy in the game \((G_{0,1},1,W)\). Indeed Pl. 0 simply has to compute the value of the counter after the current prefix and to output enough 0s to get the counter value to zero. However Pl. 0 has no bounded winning strategy or move-counting winning strategy. Assume that Pl. 0 plays according to a \( b \)-bounded strategy \( f \), then if Pl. 1 plays \( 1^{b+1} \) in each round, the counter can never reach 0. Assume that Pl. 0 plays according to a move-counting strategy \( f \), then for all \( n \), let \( b_n = |f(1,n)| \). Pl. 1 has a move-counting strategy \( g \) to win against \( f \). It suffices to consider a strategy \( g \) consisting of playing a block of \( 1 \) of length strictly larger than the length of the next move of Pl. 0. For example, such a strategy can be defined by \( g(v,n) = 1^{b_n+1} \), for all \( v \in [0,1] \) and any \( n \in \mathbb{N} \). This way, the counter can never reach 0.
Definition 4.1. A generalised Banach–Mazur game $\mathcal{G}$ on a finite graph is a tuple $(G, v_0, \phi_0, \phi_1, W)$ where $G = (V, E)$ is a finite directed graph where every vertex has a successor, $v_0 \in V$ is the initial state, $W \subseteq \text{Paths}(G, v_0)$, and $\phi_i$ is a map on $\text{FinPaths}(G, v_0)$ such that for all $\pi \in \text{FinPaths}(G, v_0)$,

$$\phi_i(\pi) \subseteq \mathcal{P}(\text{FinPaths}(G, \text{last}(\pi))) \setminus \{\emptyset\} \text{ and } \phi_i(\pi) \neq \emptyset.$$  

A generalised Banach–Mazur game $\mathcal{G} = (G, v_0, \phi_0, \phi_1, W)$ on a finite graph is a two-player game where Pl. 0 and Pl. 1 alternate in choosing sets of finite paths as follows: Pl. 1 begins with choosing a set of finite paths $\Pi_1 \in \phi_1(v_0)$; Pl. 0 selects a finite path $\pi_1 \in \Pi_1$ and chooses a set of finite paths $\Pi_2 \in \phi_0(\pi_1)$; Pl. 1 then selects $\pi_2 \in \Pi_2$ and proposes a set $\Pi_3 \in \phi_1(\pi_1, \pi_2)$ and so on. A play of $\mathcal{G}$ is thus an infinite path $\pi_1, \pi_2, \pi_3 \ldots$ in $G$ and we say that Pl. 0 wins if this path belongs to $W$, while Pl. 1 wins if this path does not belong to $W$.

We remark that if we let

$$\phi_{\text{ball}}(\pi) := \{\pi' \in \text{FinPaths}(G, \text{last}(\pi))\}$$  

for all $\pi \in \text{FinPaths}(G, v_0)$, then the generalised Banach–Mazur game $(G, v_0, \phi_{\text{ball}}, \phi_{\text{ball}}, W)$ coincides with the classical Banach–Mazur game $(G, v_0, W)$. On the other hand, if we consider $\phi(\pi) := \{\pi' \in \text{FinPaths}(G, \text{last}(\pi)) : |\pi'| = 1\}$, we obtain the classical games on graphs such as the ones studied in [7].

We are interested in defining a map $\phi_0$ such that Pl. 0 has a winning strategy in the game $(G, v_0, \phi_0, \phi_{\text{ball}}, W)$ if and only if $P(W) = 1$. To this end, we notice that we can restrict actions of Pl. 0 by forcing each set in $\phi_0(\pi)$ to be “big” in some sense. The idea to characterise $P(W) = 1$ is therefore to force Pl. 0 to play with finite sets of finite paths of conditional probability bigger than $\alpha$ for some $\alpha > 0$.

Definition 4.2. Let $\mathcal{G} = (G, v_0, W)$ be a Banach–Mazur game on a finite graph, $P$ a reasonable probability measure and $\alpha > 0$. An $\alpha$-strategy of Pl. 0 for $\mathcal{G}$ is a strategy of Pl. 0 for the generalised Banach–Mazur game $\mathcal{G}_\alpha = (G, v_0, \phi_\alpha, \phi_{\text{ball}}, W)$ where

$$\phi_\alpha(\pi) = \left\{\Pi \subseteq \text{FinPaths}(G, \text{last}(\pi)) : P\left(\bigcup_{\pi' \in \Pi} \text{Cyl}(\pi \pi') | \text{Cyl}(\pi)\right) \geq \alpha \text{ and } \Pi \text{ is finite}\right\}.$$  

We recall that, given two events $A, B$ with $P(B) > 0$, the conditional probability $P(A|B)$ is defined as $P(A|B) := P(A \cap B)/P(B)$.

We point out that the notion of $\alpha$-strategy can depend on the probability measure $P$ that we consider. However, we do not make this dependency explicit to avoid cluttering the notation.

We notice that every bounded strategy can be seen as an $\alpha$-strategy for some $\alpha > 0$, since for all $N \geq 1$, there exists $\alpha > 0$ such that for all $\pi$ of length less than $N$, we have $P(|\pi|) \geq \alpha$. We can also show that the existence of a move-counting winning strategy for Pl. 0 implies the existence of a winning $\alpha$-strategy for Pl. 0 for every $0 < \alpha < 1$. 

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Proposition 4.3. Let $G = (G, v_0, W)$ be a Banach–Mazur game on a finite graph. If $Pl. 0$ has a move-counting winning strategy, then $Pl. 0$ has a winning $\alpha$-strategy for every $0 < \alpha < 1$.

Proof. Let $P$ be a reasonable probability measure, $h$ a move-counting winning strategy for $Pl. 0$ and $0 < \alpha < 1$. By Theorem 2.8, there exists a family of positional strategies $(g_n)_{n \in \mathbb{N}}$ such that for all clock $c$, $(g_n)_{n \in \mathbb{N}}, c$ is winning. In particular, for all increasing sequence $(n_k)$, a play of the form

$$\pi_1 g_{n_1} (\text{last}(\pi_1)) \pi_2 g_{n_2} (\text{last}(\pi_2)) \cdots \pi_k g_{n_k} (\text{last}(\pi_k)) \cdots$$

is winning. Since $g_n$ is a positional strategy, we know that each set

$$M_n := \{ \rho \in \text{Paths}(G, v_0) : \rho \text{ is a play consistent with } g_n \}$$

has probability 1. In particular, for all $\pi_0 \in \text{FinPaths}(G, v_0)$, we deduce that $P(M_n | \text{Cyl}(\pi_0)) = 1$. Since

$$M_n \cap \text{Cyl}(\pi_0) \subseteq \bigcup_{\pi \in \text{FinPaths}(G, \text{last}(\pi_0))} \text{Cyl}(\pi_0 \pi g_n (\text{last}(\pi))),$$

we have

$$P\left( \bigcup_{\pi \in \text{FinPaths}(G, \text{last}(\pi_0))} \text{Cyl}(\pi_0 \pi g_n (\text{last}(\pi))) \bigg| \text{Cyl}(\pi_0) \right) = 1$$

and since $\text{FinPaths}(G, \text{last}(\pi_0))$ is countable, we deduce that for all $n \geq 1$, any $\pi_0 \in \text{FinPaths}(G, v_0)$, there exists a finite subset $\Pi_n(\pi_0) \subset \text{FinPaths}(G, \text{last}(\pi_0))$ such that

$$P\left( \bigcup_{\pi \in \Pi_n(\pi_0)} \text{Cyl}(\pi_0 \pi g_n (\text{last}(\pi))) \bigg| \text{Cyl}(\pi_0) \right) \geq \alpha.$$

We denote by $\Pi_n(\pi_0)$ the set $\{ \pi g_n (\text{last}(\pi)) : \pi \in \Pi_n(\pi_0) \}$ and we let

$$f(\pi_0) := \Pi_{\lfloor \max(\pi_0) \rfloor}(\pi_0).$$

The above-defined strategy $f$ is therefore a winning $\alpha$-strategy for $Pl. 0$ since each play consistent with $f$ is of the form (4.1) for some sequence $(n_k)$ and thus winning. \( \Box \)

Moreover, the existence of a winning $\alpha$-strategy for some $\alpha > 0$ still implies $P(W) = 1$.

Theorem 4.4. Let $G = (G, v_0, W)$ be a Banach–Mazur game on a finite graph and $P$ a reasonable probability measure. If $Pl. 0$ has a winning $\alpha$-strategy for some $\alpha > 0$, then $P(W) = 1$.

Proof. Let $f$ be a winning $\alpha$-strategy. We consider an increasing sequence $(a_n)_{n \geq 1}$ such that for all $n \geq 1$, any $\pi$ of length $a_n$, each $\pi' \in f(\pi)$ has length less than $a_{n+1} - a_n$; this is possible because for all $\pi$, $f(\pi)$ is a finite set by definition of $\alpha$-strategies. Without loss of generality, we can even assume that for all $n \geq 1$, any $\pi$ of length $a_n$, each $\pi' \in f(\pi)$ has exactly length $a_{n+1} - a_n$. Therefore we let

$$A := \{ (\sigma_k)_{k \geq 1} \in \text{Paths}(G, v_0) : \#\{ n : (\sigma_k)_{a_{n+1} \leq k \leq a_n} \in f(\sigma_k) \} \leq \infty \}.$$

In other words, $(\sigma_k)_{k \geq 1} \in A$ if $(\sigma_k)$ can be seen as a play where $f$ has been played on an infinite number of indices $a_n$. Since $f$ is a winning strategy, $A$ is included in $W$ and it thus suffices to prove that $P(A) = 1$.

We first notice that for all $m \geq 1$, all $n \geq m$, if we let

$$B_{m,n} = \{ (\sigma_k)_{k \geq 1} \in \text{Paths}(G, v_0) : (\sigma_k)_{a_j+1 \leq k \leq a_{j+1}} \not\in f(\sigma_k)_{1 \leq k \leq a_j}, \forall m \leq j \leq n \},$$

then $P(B_{m,n}) \leq (1 - \alpha)^{n+1-m}$ as $f$ is an $\alpha$-strategy. We therefore deduce that for all $m \geq 1$,

$$P\left( \bigcap_{n=m}^{\infty} B_{m,n} \right) = 0$$

and since $A^c = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} B_{m,n}$, we conclude that $P(A) = 1$. \( \Box \)

If $W$ is a countable intersection of open sets, we can prove the converse of Theorem 4.4 and so obtain a characterisation of sets of probability 1.

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5 Let $\pi$ be a finite path and $n_\pi \geq \max(\|\pi\| : \text{such that } \tau \in f(\pi))$. One can define $f(\pi)$ as the set of finite paths $\sigma$ of length $n_\pi$ such that $\tau$ is a prefix of $\sigma$, for some $\tau \in f(\pi)$. Given a play $\rho$, one can show that $\rho$ is consistent with $f$ if and only if $\rho$ is consistent with $f$. Please cite this article in press as: T. Brihaye et al., Simple strategies for Banach-Mazur games and sets of probability 1. Inf. Comput. (2015). http://dx.doi.org/10.1016/j.ic.2015.06.004
Theorem 4.5. Let \( G = (G, v_0, W) \) be a Banach–Mazur game on a finite graph where \( W \) is a countable intersection of open sets and \( P \) a reasonable probability measure. Then the following assertions are equivalent:

1. \( P(W) = 1 \).
2. Pl. 0 has a winning \( \alpha \)-strategy for some \( \alpha > 0 \).
3. Pl. 0 has a winning \( \alpha \)-strategy for all \( 0 < \alpha < 1 \).

Proof. We have already proved 2. \( \Rightarrow \) 1., and 3. \( \Rightarrow \) 2. is obvious.

1. \( \Rightarrow \) 3. Let \( 0 < \alpha < 1 \). Let \( W = \cap_{n=1}^\infty W_n \) where the \( W_n \)'s are open sets. Since \( P(W) = 1 \), we deduce that for all \( n \geq 1 \), \( P(W_n) = 1 \). We can therefore define a winning \( \alpha \)-strategy \( f \) of Pl. 0 as follows: if \( Cyl(\pi) \subset \cap_{k=1}^{n-1} W_k \) and \( Cyl(\pi) \not\subset W_n \), we let \( f(\pi) \) be a finite set \( \Pi \subset \text{FinPaths}(G, \text{last}(\pi)) \) such that \( P\left( \bigcup_{\pi' \in \Pi} Cyl(\pi \pi') \setminus Cyl(\pi) \right) \geq \alpha \) and for all \( \pi' \in \Pi \), \( Cyl(\pi \pi') \subset W_n \). Such a finite set \( \Pi \) exists because \( W_n \) has probability 1 and \( W_n \) is an open set, i.e. a countable union of cylinders. This concludes the proof. \( \Box \)

Remark 4.6. We cannot hope to generalise the latter result to the whole class of sets. More precisely, there exist sets of probability 1 for which no winning \( \alpha \)-strategy exists. Indeed, given a set \( W \), on the one hand, the existence of a winning \( \alpha \)-strategy for \( W \) implies the existence of a winning strategy for \( W \), and thus in particular such a \( W \) is large. On the other hand, we know that there exists some meagre (in particular not large) set of probability 1 (see Example 3.2). However, one can ask whether the existence of a winning \( \alpha \)-strategy is equivalent to the fact that \( W \) is a large set of probability 1.

When \( W \) is a countable intersection of open sets, we remark that the generalised Banach–Mazur game \( G_\alpha = (G, v_0, \phi_\alpha, \phi_{ball}, W) \) is in fact determined.

Theorem 4.7. Let \( G_\alpha \) be the generalised Banach–Mazur game given by \( G_\alpha = (G, v_0, \phi_\alpha, \phi_{ball}, W) \) where \( G \) is a finite graph, \( W \) is a countable intersection of open sets and \( P \) a reasonable probability measure. Then the following assertions are equivalent:

1. \( P(W) < 1 \).
2. Pl. 1 has a winning strategy for \( G_\alpha \) for some \( \alpha > 0 \).
3. Pl. 1 has a winning strategy for \( G_\alpha \) for all \( 0 < \alpha < 1 \).

Proof. We deduce from Theorem 4.5 that 2. \( \Rightarrow \) 1. because \( G_\alpha \) is a zero-sum game, and 3. \( \Rightarrow \) 2. is obvious.

1. \( \Rightarrow \) 3. Let \( W = \cap_{n=1}^\infty W_n \) with \( P(W) < 1 \) and \( W_n \) open. We know that there exists \( n \geq 1 \) such that \( P(W_n) < 1 \). It then suffices to prove that Pl. 1 has a winning strategy for the generalised Banach–Mazur game \( G, v_0, \phi_\alpha, \phi_{ball}, W_n \) for all \( 0 < \alpha < 1 \). Without loss of generality, we can thus assume that \( W \) is an open set. We recall that \( W \) is open if and only if it is a countable union of cylinders. Since any strategy of Pl. 1 is winning if \( W = \emptyset \), we also suppose that \( W \neq \emptyset \).

Let \( 0 < \alpha < 1 \). We first show that there exists a finite path \( \pi_1 \in \text{FinPaths}(G, v_0) \) such that any set \( \Pi_2 \in \phi_\alpha(\pi_1) \) contains a finite path \( \pi_2 \) satisfying

\[
P(W|Cyl(\pi_1 \pi_2)) \leq P(W) < 1. \tag{4.2}
\]

Let

\[
I_W := \inf\{P(W|Cyl(\pi)) : \pi \in \text{FinPaths}(G, v_0)\}. \tag{4.3}
\]

Since \( W \) is a non-empty union of cylinders, there exists \( \sigma \in \text{FinPaths}(G, v_0) \) such that we have \( P(W|Cyl(\sigma)) = 1 \). We remark that

\[
P(W) = \sum_{\pi : |\pi| = |\sigma|} P(W|Cyl(\pi))P(Cyl(\pi)) \quad \text{and} \quad \sum_{\pi : |\pi| = |\sigma|} P(Cyl(\pi)) = 1.
\]

Therefore, since \( P(W|Cyl(\sigma)) > P(W) \), we deduce that there exists \( \pi \in \text{FinPaths}(G, v_0) \) with \( |\pi| = |\sigma| \) such that \( P(W|Cyl(\pi)) < P(W) \) and we conclude that \( I_W < P(W) \). Moreover, in view of the definition of \( I_W \), we know that for any \( \varepsilon > 0 \), there exists \( \pi \in \text{FinPaths}(G, v_0) \) such that \( P(W|Cyl(\pi)) - I_W < \varepsilon \). There thus exists \( \pi_1 \in \text{FinPaths}(G, v_0) \) such that

\[
I_W + \frac{1}{\alpha}(P(W|Cyl(\pi_1)) - I_W) < P(W). \tag{4.4}
\]

Let \( \Pi_2 \in \phi_\alpha(\pi_1) \). We consider \( \tau_1, \ldots, \tau_n \in \Pi_2 \times \Pi_1 \) and \( \sigma_1, \ldots, \sigma_m \in \text{FinPaths}(G, \text{last}(\pi_1)) \) such that cylinders \( Cyl(\tau_i) \), \( Cyl(\sigma_j) \) are pairwise disjoint, \( \bigcup_{\tau \in \Pi_2} Cyl(\tau) \subset \bigcup_{i=1}^n Cyl(\tau_i) \) and

\[
\text{Paths}(G, \text{last}(\pi_1)) = \bigcup_{i=1}^n Cyl(\tau_i) \bigcup_{j=1}^m Cyl(\sigma_j). \tag{4.5}
\]
Assume that for all $1 \leq i \leq n$, we have
\[ P(W|\text{Cyl}(\tau_1)) > P(W). \]
(4.6)

Then, we get
\[
P(W|\text{Cyl}(\tau_1)) \\
= \sum_{i=1}^{n} P(W \cap \text{Cyl}(\tau_1))|\text{Cyl}(\tau_1)) + \sum_{j=1}^{m} P(W \cap \text{Cyl}(\tau_j))|\text{Cyl}(\tau_1)) \text{ by disjointedness and (4.5)} \\
= \sum_{i=1}^{n} P(W|\text{Cyl}(\tau_1))P(\text{Cyl}(\tau_1)|\text{Cyl}(\tau_1)) + \sum_{j=1}^{m} P(W|\text{Cyl}(\tau_j))P(\text{Cyl}(\tau_j)|\text{Cyl}(\tau_1)) \\
\geq P(W) \sum_{i=1}^{n} P(\text{Cyl}(\tau_1)|\text{Cyl}(\tau_1)) + I_W \sum_{j=1}^{m} P(\text{Cyl}(\tau_j)|\text{Cyl}(\tau_1)) \text{ by (4.6) and (4.3)} \\
\geq P(W) \sum_{i=1}^{n} P(\text{Cyl}(\tau_1)|\text{Cyl}(\tau_1)) + I_W (1 - \sum_{i=1}^{n} P(\text{Cyl}(\tau_1)|\text{Cyl}(\tau_1))) \text{ by (4.5)} \\
\geq P(W) \left( \bigcup_{\pi \in \Pi_2} \text{Cyl}(\pi) \right) + I_W (1 - \left( \bigcup_{\pi \in \Pi_2} \text{Cyl}(\pi) \right)) \text{ by properties of } \tau_1, \text{ and}\n
\geq P(W) \alpha + I_W (1 - \alpha) \quad (\text{because } \Pi_2 \in \phi_\alpha(\tau_1) \text{ and } P(W) > I_W)
\]
and thus $P(W) \leq I_W + \frac{1}{\alpha}(P(W|\text{Cyl}(\tau_1)) - I_W)$ which is a contradiction with (4.4). We conclude that if $\tau_1$ is given as in (4.4), then any set $\Pi_2 \in \phi_\alpha(\tau_1)$ contains a finite path $\tau_2$ satisfying (4.2).

We can now exhibit a winning strategy for Pl. 1. We assume that Pl. 1 begins with playing a finite path $\tau_1$ satisfying (4.4). Let $f$ be an $\alpha$-strategy. We know that Pl. 1 can select a finite path $\tau_2 \in f(\tau_1)$ satisfying (4.2), i.e. $P(W|\text{Cyl}(\tau_1, \tau_2)) \leq P(W)$. By repeating the above method from $\tau_1, \tau_2$, we deduce the existence of a finite path $\tau_3$ such that any set $\Pi_4 \in \phi_\alpha(\tau_1, \tau_2, \tau_3) \text{ is a finite path } \tau_4$ satisfying $P(W|\text{Cyl}(\tau_1, \tau_2, \tau_3, \tau_4)) \leq P(W)$. We can thus assume that Pl. 1 plays such a finite path $\tau_3$ and then selects $\tau_4 \in f(\tau_3)$ such that $P(W|\text{Cyl}(\tau_1, \tau_2, \tau_3, \tau_4)) \leq P(W)$. This strategy is a winning strategy for Pl. 1. Indeed, as $W$ is an open set and thus a countable union of cylinders, if $P(W|\text{Cyl}(\tau_1, \cdots, \tau_2n)) \leq P(W) < 1$ for all $n$, then $\tau_1, \tau_2, \tau_3, \cdots \notin W$. □

**Corollary 4.8.** Let $0 < \alpha < 1$. The generalised Banach–Mazur game $G_{\alpha} = (G, v_0, \phi_\alpha, \phi_{ball}, W)$ is determined when $W$ is a countable intersection of open sets. More precisely, Pl. 0 has a winning strategy for $G_{\alpha}$ if and only if $P(W) = 1$, and Pl. 1 has a winning strategy for $G_{\alpha}$ if and only if $P(W) < 1$.

Since the existence of a bounded winning strategy for Pl. 0 implies the existence of a winning $\alpha$-strategy for Pl. 0 and the existence of a move-counting winning strategy for Pl. 0 implies the existence of a winning $\alpha$-strategy for Pl. 0, we deduce from Example 2.6 and Example 2.7 that in general, the existence of a winning $\alpha$-strategy for Pl. 0 implies neither the existence of a move-counting winning strategy for Pl. 0 nor the existence of a bounded winning strategy for Pl. 0. On the other hand, we know that there exists a Banach–Mazur game for which Pl. 0 has a bounded winning strategy and no last-move winning strategy. The existence of a winning $\alpha$-strategy thus does not imply in general the existence of a last-move winning strategy. Conversely, if we consider the game $(G_{0,1}, 0, W)$ described in Example 3.2, Pl. 0 has a last-move winning strategy but no winning $\alpha$-strategy (as $P(W) = 0$). The notion of $\alpha$-strategy is thus incomparable with the notion of last-move strategy.

**5. More on simple strategies on finite graphs**

We finish this section by investigating other notions of simple strategies on finite graphs. First, we consider memory strategies, i.e. strategies restricted by memory machines, and we discuss the size of the memory needed to keep the full power of length-counting strategies (which have the same expressive power as the un restricted strategies). Since finite-memory strategies are strictly less powerful than length-counting strategies, we know that we need to have infinitely many memory states. Therefore we study how the used number of states evolve with the size of the prefix constructed. More precisely given a game $(G, v_0, W)$ and a memory machine $m: \text{FinPaths}(G, v_0) \rightarrow S$, we define the mapping $\#_m: \mathbb{N} \rightarrow \mathbb{N}$ by setting
\[
\#_m(n) = \left| \{ s \in S \mid \exists \pi \mid |\pi| \leq n \land m(v_0, \pi) = s \} \right|.
\]
i.e. \( \#_m(n) \) is the number of possible memory states reached after having constructed a finite path of length at most \( n \). For example considering the length-counting memory machine \( m_{lc} \) we have for all \( n \), \( \#_{m_{lc}}(n) = n \). On the graph \( G_0,1 \), i.e. the complete graph on \([0, 1]\), considering the move-blind memory machine \( m_{mb} \), we have \( \#_{m_{mb}}(n) = 2^n \) for all \( n \).

The mapping \( \#_m \), called the memory complexity, is a good description of the evolution of the size of the memory tape of a memory machine. The fact that length-counting winning strategies are as powerful as unrestricted strategies shows that a linear memory complexity is enough to win any game where it is possible to win. One can see that this is a tight restriction, as described in the following theorem.

**Proposition 5.1.** There exists a Banach–Mazur game such that \( Pl_0 \) has a winning strategy but \( Pl_0 \) has no \( m \)-restricted winning strategy for all memory machines \( m \) such that \( \lim \sup_{n \to \infty} (n - \#_m(n)) = \infty \).

**Proof.** Let \((G_0,1,0,W)\) be the game of Example 3.3, i.e. the game on the complete two vertices graph, such that an infinite path \( w \) belongs to \( W \) if and only if there exists \( i \) such that the \( a_i \)th vertex \( w_i \) of \( w \) is 1, where \( a_i = \sum_{k=1}^{n} k \). As seen previously there exists a winning strategy for \( Pl_0 \), basically outputting enough 1’s to reach \( a_i \) for some \( i \). Now let \( m \) be a memory machine such that \( \lim \sup_{n \to \infty} (n - \#_m(n)) = \infty \) and \( f \) an \( m \)-restricted strategy. Without loss of generality we can assume that \( Pl_0 \) always outputs some words from \( 1^* \) and \( Pl_1 \) always outputs some words from \( 0^* \).

First observe that since \( \lim \sup_{n \to \infty} (n - \#_m(n)) = \infty \), for all infinite path \( w = v_1v_2 \cdots \) for all \( i \), there exists \( j < k < n \) such that \( m(0,v_1 \cdots v_j) = m(0,v_1 \cdots v_k) \). Indeed otherwise for all \( j,k \), such that \( i < j < k \), we would have \( m(0,v_1 \cdots v_j) \neq m(0,v_1 \cdots v_k) \). In particular, this would imply that for all \( n \geq i \), \( \#_m(n) \geq n - i \), and thus \( \lim \sup_{n \to \infty} (n - \#_m(n)) \leq i \). We can therefore define a strategy \( g \) for \( Pl_1 \) that wins against \( f \).

Intuitively, \( Pl_1 \) will reach a repetion of a memory state along a finite sequence of 0’s, and will iterate this sequence long enough to ensure that on the next move, \( Pl_0 \) will not put 1 on an \( a_i \)th position.

Given a finite path \( \pi \), let \( j, \ell \in \mathbb{N} \) be such that \( m(0,\pi 0^j) = m(0,\pi 0^{j+\ell}) \). From the previous remark we know that such \( j \) and \( \ell \) exist. Let \( s = m(0,\pi 0^j) = m(0,\pi 0^{j+\ell}) + 1 \) and \( f = f(0,s) \). Since \( m \) is a memory machine, for all \( d \geq 1 \), \( m(0,\pi 0^j) = m(0,\pi 0^{j+dl}) \) and since \( (a_{i+1} - a_i)_{i \in \mathbb{N}} \) is a strictly increasing sequence, we know that there exist \( d \geq 1 \) and an index \( i \), such that:

\[
a_i < |\pi| + j + d \ell < |\pi| + j + d \ell + n < a_{i+1}.
\]

By following the strategy \( g(\pi) = 0^{j+dl} \), \( Pl_1 \) therefore ensures that \( Pl_0 \) never puts a 1 on an \( a_i \)th position. \( \square \)

We finish this section by considering the crossings between the classical notions of simple strategies and the notion of bounded strategy, i.e. the bounded length-counting strategies, the bounded move-counting strategies and the bounded last-move strategies. Obviously, the existence of a bounded length-counting winning strategy for \( Pl_0 \) implies the existence of a length-counting winning strategy for \( Pl_0 \), and we have this implication for each notion of bounded strategies and their non-bounded counterpart. We start by noticing that the existence of a bounded move-counting winning strategy is equivalent to the existence of a positional winning strategy.

**Proposition 5.2.** Let \( G = (G,v_0, W) \) be a Banach–Mazur game on a finite graph. \( Pl_0 \) has a bounded move-counting winning strategy if and only if \( Pl_0 \) has a positional winning strategy.

**Proof.** Let \( h \) be a bounded move-counting winning strategy for \( Pl_0 \). We denote by \( C_1, \ldots, C_N \) the bottom strongly connected components (BSCC) of \( G \). Let \( 1 \leq i \leq N \). Since \( h \) is a bounded strategy and \( G \) is finite, there exist some finite paths \( w_1^{(i)}, \ldots, w_k^{(i)} \subset C_i \) such that for all \( v \in C_i \), for all \( n \geq 1 \),

\[
h(v,n) \in \{w_1^{(i)}, \ldots, w_k^{(i)}\}.
\]

Let \( v \in V \). If \( v \in C_i \), we let \( f(v) = \sigma_0 w_1^{(i)} \sigma_1 w_2^{(i)} \sigma_2 \ldots w_k^{(i)} \) where \( \sigma_i \) are finite paths in \( C_i \) such that \( f(v) \) is a finite path in \( C_i \) starting from \( v \). If \( v \notin \bigcup C_i \), we let \( f(v) = \sigma_v \) where \( \sigma_v \) starts from \( v \) and leads into a BSCC of \( G \). The positional strategy \( f \) is therefore winning as each play \( \rho \) consistent with \( f \) can be seen as a play consistent with \( h \). \( \square \)

The other notions of bounded strategies are not equivalent to any other notion of simple strategy considered here.

**Example 5.3** (Set with a bounded length-counting winning strategy and without a positional winning strategy). Let \( G_{0,1} \) be the complete graph on \([0,1]\), \( (\rho_a) \) an enumeration of finite words in \([0,1]\) and \( \rho_{\text{target}} = 0\rho_1\rho_2 \cdots \). We consider the set \( W = \{ \sigma \in \{0,1\}^\omega : |\sigma| \geq 1 : \sigma(i) = \rho_{\text{target}}(i) = \infty \} \). It is evident that \( Pl_0 \) has a bounded length-counting winning strategy for the game \((G_{0,1},0,W) \). However, \( Pl_0 \) has no positional winning strategy. Indeed, if \( f \) is a positional strategy such that \( f(0) = \sigma(1) \cdots \sigma(k) \), then \( Pl_1 \) can play according to the strategy \( h \) defined by \( h(\sigma(1) \cdots \sigma(n)) = \sigma(n+1) \cdots \sigma(N) \) such that for all \( n + 1 \leq i \leq N \), \( \sigma(i) \neq \rho_{\text{target}}(i) \), \( \rho_{\text{target}}(N+1) \neq 0 \) and for all \( 1 \leq i \leq k \), \( a(i) \neq \rho_{\text{target}}(N+i+1) \).
Example 5.4 (Set with a bounded last-move winning strategy and without a positional winning strategy). Let $G_{0,1,2}$ be the complete graph on $[0,1,2]$. For all $\phi : [0,1,2]^* \rightarrow [0,1]$, if we consider the set $W := \{\pi(\phi(\pi))_{i \geq 1} : \pi_i \in [0,1,2]^*\}$, then Pl. 0 has a 1-bounded last-move winning strategy given by $\phi$ for the game $(G_{0,1,2}, 2, W)$. On the other hand, we can choose $\phi$ such that Pl. 0 has no positional winning strategy. Indeed, it suffices to choose $\phi : [0,1,2]^* \rightarrow [0,1]$ such that for all $\pi \in [0,1,2]^*$, any $n \geq 1$, any $\sigma(1), \ldots, \sigma(n) \in [0,1,2]$, there exists $k \geq 1$ such that $\phi(\pi 2^k \sigma(1) \cdots \sigma(i)) \neq \sigma(i+1)$. Such a function exists because the sets $[0,1,2]^*$ and $[2^k : k \geq 1]$ are countable. Therefore, Pl. 0 has no positional winning strategy for the game $(G_{0,1,2}, 2, W)$ because, if $f$ is a positional strategy and $f(2) = \sigma(1) \cdots \sigma(n)$, then Pl. 1 can play consistent with the strategy $h$ defined by $h(\pi) = 2^k$ such that $\phi(\pi 2^k) \neq \sigma(1)$ and for any $1 \leq i \leq n - 1$, $\phi(\pi 2^k \sigma(1) \cdots \sigma(i)) \neq \sigma(i+1)$. Pl. 0 has thus a 1-bounded last-move winning strategy and no positional winning strategy for the game $(G_{0,1,2}, 2, W)$.

Example 5.5 (Set with a bounded winning strategy and without a bounded length-counting winning strategy). Let $G_{0,1,2,3}$ be the complete graph on $[0,1,2,3]$ and $\phi$ a map from $[0,1,2,3]^*$ to $[0,1]$. If $W$ is the set of runs $\rho$ such that $\#[n \geq 1 : \rho(1) \cdots \rho(n)] = \rho(n+1) = \infty$, then Pl. 0 has a 1-bounded winning strategy given by $\phi$ for the game $(G_{0,1,2,3}, 2, W)$. We now show how we can define $\phi$ so that Pl. 0 has no bounded length-counting winning strategy. Let $n_k = \sum_{i=1}^k 3i$. We choose $\phi : [0,1,2,3]^* \rightarrow [0,1]$ such that for all $k \geq 1$, any $\pi \in [0,1,2,3]^*$ of length $n_k$ and any $\sigma(1), \ldots, \sigma(k) \in [0,1,2,3]$, there exists $\tau \in [2,3]^k$ of length $2k$ such that $\phi(\tau \sigma(1) \cdots \sigma(i)) \neq \sigma(i+1)$. Such a function exists because the cardinality of $[2,3]^k$ is equal to the cardinality of $[0,1,2,3]^k$ and the length of $\pi \tau \sigma(1) \cdots \sigma(k) < n_{k+1}$. Therefore, Pl. 0 has no bounded length-counting winning strategy because if $f$ is a $k$-bounded length-counting strategy (for some $k \in \mathbb{N}$) and $f(2, n_k + k + 1) = \sigma$, then Pl. 1 can start by playing $2^k \tau \rho$, where $\tau \in [2,3]^k$ of length $2k$ such that $\phi(\tau \rho) \neq \sigma(1)$ and for all $1 \leq i \leq k$, $\phi(\tau \rho \sigma(1) \cdots \sigma(i)) \neq \sigma(i+1)$, and if Pl. 1 keeps playing with same philosophy, then Pl. 1 wins the play. Pl. 0 has thus a 1-bounded winning strategy and no bounded length-counting winning strategy for the game $(G_{0,1,2,3}, 2, W)$.

The relations between the different notions of simple strategies on a finite graph can be summarised as depicted in Fig. 3. We draw attention to the fact that the situation is very different in the case of infinite graphs. For example, a positional strategy can be unbounded, the notion of length-counting winning strategy is not equivalent to the notion of winning strategy (except if the graph is finitely branching), and the notion of bounded move-counting winning strategy for Pl. 0 is not equivalent to the notion of positional winning strategy.

Example 5.6 (Set on an infinite graph with a bounded move-counting winning strategy and without a positional winning strategy). We consider the complete graph $G_\mathbb{N}$ on $\mathbb{N}$ and the game $G = (G_\mathbb{N}, 0, W)$ where $W = \{\sigma_k \in [0,\infty] : \forall n \geq 1, \exists k \geq 1, (\sigma_k, \sigma_{k+1}) = (n, n+1)\}$. Pl. 0 has a bounded move-counting winning strategy given by $h(v,n) = n + 1$ but no positional winning strategy.
6. Banach–Mazur games on topological spaces

6.1. General background

Banach–Mazur games on finite graphs are a special case of Banach–Mazur games on topological spaces where two players alternate in choosing open sets in an open basis. We recall that if $X$ is a topological space, an open basis $\mathcal{V}$ is a family of open sets in $X$ such that every open set in $X$ can be written as a union of elements of $\mathcal{V}$. In particular, every non-empty open set in $X$ contains a non-empty element of $\mathcal{V}$.

Definition 6.1. A Banach–Mazur game $\mathcal{G}$ on a topological space is a triplet $(X, \mathcal{V}, W)$ where $X$ is a topological space, $\mathcal{V}$ is an open basis and $W \subset X$.

A Banach–Mazur game $\mathcal{G} = (X, \mathcal{V}, W)$ is a two-player game where Pl. 0 and Pl. 1 alternate in choosing a non-empty open set in $\mathcal{V}$ as follows: Pl. 1 begins with choosing a non-empty open set $O_1 \in \mathcal{V}$; Pl. 0 then chooses a non-empty open set $O_2 \subset O_1$ in $\mathcal{V}$; afterwards, Pl. 1 chooses a non-empty open set $O_3 \subset O_2$ in $\mathcal{V}$ and so on. A play of $\mathcal{G}$ is thus a decreasing sequence of non-empty open sets $(O_n) \subset \mathcal{V}$ and we say that Pl. 0 wins if $\bigcap_n O_n \cap W = \emptyset$; while Pl. 1 wins if $\bigcap_n O_n \cap W = \emptyset$. The set $W$ is called the winning condition.

If $X$ is a metric space, we denote by $\mathcal{G} = (X, W)$ the Banach–Mazur game $(X, \mathcal{V}, W)$ where $\mathcal{V}$ is the set of open balls in $X$. In particular, if we consider the metric space $(\text{Paths}(\mathcal{G}, v_0), d)$ where $d$ is given by (2.1), then the Banach–Mazur game $(\text{Paths}(\mathcal{G}, v_0), W)$ coincides with the Banach–Mazur game on the finite graph $G$ since the open balls for the metric $d$ are given by the cylinders in $G$ starting from $v_0$.

In the case of finite graphs, we know that the largeness of a set $W$ is equivalent to the existence of a winning strategy of Pl. 0 for the related Banach–Mazur game on a finite graph (Theorem 2.2). We can in fact generalise this correspondence to complete metric spaces. We recall that a set $A \subset X$ is said to be nowhere dense if the closure of $A$ has empty interior, a set $A \subset X$ is said to be meagre if it can be expressed as the union of countably many nowhere dense sets and a set $A \subset X$ is said to be large if the complement of $A$ is meagre.

Theorem 6.2. (See [10].) Let $\mathcal{G} = (X, W)$ be a Banach–Mazur game where $X$ is a complete metric space.

1. Pl. 0 has a winning strategy for $\mathcal{G}$ if and only if $W$ is large.

2. Pl. 1 has a winning strategy for $\mathcal{G}$ if and only if $W$ is meagre in some non-empty open set.

The assumptions of completeness and metrisability are very important. Indeed, if we consider the metric space given by $(\mathbb{Q}, | \cdot |)$ then every set $W \subset \mathbb{Q}$ is large but Pl. 1 has a winning strategy for all Banach–Mazur game $(\mathbb{Q}, W)$. On the other hand, if we consider the trivial topology on a set $X$, (i.e. $X$ is the only non-empty open set), then no subset $W \neq X$ is large and Pl. 0 has a winning strategy for all Banach–Mazur game $(X, W)$.

We remark that for Banach–Mazur games on topological spaces, several notions of simple strategies are not well defined. Nevertheless, we can generalise the notion of $\alpha$-strategy to this context and we will show in the remainder of this paper that the obtained results for $\alpha$-strategies in the previous section can be extended to separable complete metric spaces.

6.2. Generalised Banach–Mazur games on topological spaces

In Section 4, we have introduced the notion of generalised Banach–Mazur games on finite graphs in order to obtain a characterisation of subsets of probability 1. Our results (Theorems 4.5 and 4.7) can in fact be extended to complete metric spaces in introducing a notion of generalised Banach–Mazur games to topological spaces. However, since the arguments used in the metric case are more technical and rely on several properties of probability measures on metric spaces, we have preferred to start with the particular case of finite graphs.

Let $X$ be a topological space and $W \subset X$. For all non-empty open set $O \subset X$, we denote by $\mathcal{O}(O)$ the set of non-empty open subsets of $O$. We can then define generalised Banach–Mazur games on topological spaces as follows:

Definition 6.3. A generalised Banach–Mazur game $\mathcal{G} = (X, \phi_0, \phi_1, W)$ is a tuple where

1. $X$ is a topological space;
2. $\phi_1$ is a map from $\mathcal{O}(X)$ to $\mathcal{P}(\mathcal{O}(X))$ such that for all $O \in \mathcal{O}(X)$, $\phi_1(O) \subset \mathcal{O}(O)$ and $\phi_1(O) \neq \emptyset$.
3. $W \subset X$ is the winning condition.

In a generalised Banach–Mazur game, Pl. 1 begins with choosing an element $O_1 \in \phi_1(X)$; Pl. 0 then chooses an element $O_2 \in \phi_0(O_1)$; afterwards Pl. 1 chooses an element $O_3 \in \phi_1(O_2)$ and so on. We say that Pl. 0 wins if and only if

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\( \cap_n O_n \cap W \neq \emptyset \); while Pl. 1 wins if and only if \( \cap_n O_n \cap W = \emptyset \). A Banach--Mazur game \((X, \mathcal{V}, W)\) is a particular case of generalised Banach--Mazur games where for all \( O \in \mathcal{O}(X) \),

\[ \phi_1(O) = \mathcal{O}(O) \cap \mathcal{V}. \]

If \((X,d)\) is a metric space, we recall that the diameter of a subset \( E \subset X \) is given by \( \text{diam}(E) = \sup \{d(x,y) : x, y \in E\} \). In the sequel, we will consider the specific function

\[ \phi_{\text{ball}}(O) = \left\{ O' \subset O : \text{diam}(O') < \infty, \text{ and } 2\text{diam}(O') \leq \text{diam}(O) \right\}. \]

**Remark 6.4.** In the case of finite graphs, we have endowed the space of infinite paths in \( G \) with the complete metric given by (2.1) and we have defined the function \( \phi_{\text{ball}} \) as follows

\[ \phi_{\text{ball}}(\pi) := \{ \pi' : \pi' \in \text{FinPaths}(G, \text{last}(\pi)) \}. \]

The general definition of \( \phi_{\text{ball}} \) is a natural generalisation of the above definition in the sense where for all \( \pi \in \text{FinPaths}(G, v_0) \), any \( \pi' \in \text{FinPaths}(G, \text{last}(\pi)) \), we have

\[ Cyl(\pi \pi') \subset Cyl(\pi) \text{ and } 2\text{diam}(Cyl(\pi \pi')) \leq \text{diam}(Cyl(\pi)). \]

Since \( O \in \mathcal{O}(O) \), the play \((O_n)_{n \geq 1}\) with \( O_n = O \) can be the outcome of a generalised Banach--Mazur game. However, it is not the case if we consider the function \( \phi_{\text{ball}} \). We can in fact show the following result.

**Proposition 6.5.** Let \( X \) be a complete metric space.

1. for all play \((O_n)_{n \geq 1}\) of \((X, \phi_0, \phi_{\text{ball}}, W)\), the intersection \( \cap_{n \geq 1} O_n \) is a singleton.
2. If \( f \) is a winning strategy of Pl. 0 for the generalised Banach--Mazur game \((X, \phi_0, \phi_{\text{ball}}, W)\) and if for all \( O \in \mathcal{O}(X) \), any \( O' \in \phi_0(O) \), any \( O'' \in \mathcal{O}(O) \), we have \( O' \cup O'' \in \phi_0(O) \), then the function \( \tilde{f} \) defined by

\[ \tilde{f}(O) = \bigcup_{O' \in \mathcal{O}(O)} f(O') \]

is a winning strategy of Pl. 0. In particular, for all non-empty open sets \( O' \subset O \), \( \tilde{f}(O') \subset \tilde{f}(O) \).

**Proof.** 1. Let \((O_n)_{n \geq 1}\) be a play of \((X, \phi_0, \phi_{\text{ball}}, W)\). We deduce from the definition of \( \phi_{\text{ball}} \) that for all \( n \geq 1 \),

\[ \text{diam}(O_{2n+1}) \leq \frac{1}{2} \text{diam}(O_{2n-1}) < \infty. \]

The set \( \cap_{n \geq 1} O_n \) is thus a singleton or the empty set. It remains to prove that \( \cap_{n \geq 1} O_n \neq \emptyset \). In fact, we also know by definition of \( \phi_{\text{ball}} \) that

\[ \cap_{n \geq 1} O_n = \cap_{n \geq 1} \overline{O_{2n+1}}. \]

Since \( X \) is complete, we therefore deduce that the set \( \cap_{n \geq 1} O_n \) is not empty. Indeed, if we consider a sequence \((x_n)_{n \geq 1}\) such that for any \( n \geq 1 \), \( x_n \in O_{2n+1} \), then the sequence \((x_n)_{n \geq 1}\) is a Cauchy sequence and there thus exists \( x \in X \) such that the sequence converges \((x_n)_{n \geq 1}\) to \( x \). We conclude that \( x \in \overline{O_{2n+1}} \) for any \( n \geq 1 \) and thus that \( x \in \cap_{n \geq 1} O_n \).

2. If \( f \) is a strategy of Pl. 0 for \((X, \phi_0, \phi_{\text{ball}}, W)\), then, by assumption on \( \phi_0 \), the function \( \tilde{f} \) defined by

\[ \tilde{f}(O) = \bigcup_{O' \in \mathcal{O}(O)} f(O') \]

is still a strategy for the game \((X, \phi_0, \phi_{\text{ball}}, W)\). Let us assume that \( f \) is winning. Let \((O_n)_{n \geq 1}\) be a play played according to \( f \). We know that \( \cap_{n \geq 1} O_n = \{x\} \) for some \( x \in X \). It remains to prove that \( x \in W \). Since \( x \in \tilde{f}(O_1) = \bigcup_{O' \in \mathcal{O}(O_1)} f(O') \), there exists \( O'_1 \in \mathcal{O}(O_1) \) such that \( x \in f(O'_1) \). Moreover, since \( f(O'_1) \) is a neighbourhood of \( x \) and \( \text{diam}(O_{2n+1}) \) tends to 0, there exists \( n_1 > 0 \) such that \( x \in O_{2n+1} \subset f(O'_1) \). We can then find \( O' \in \mathcal{O}(O_{2n+1}) \) such that \( x \in f(O') \).

By repeating this argument we construct a sequence of non-empty open sets \( O'_k \) and an increasing sequence \((n_k)_{k \geq 0}\) with \( n_0 = 0 \) such that the sequence given by \( O'_1 \), \( f(O'_1) \), \( O'_2 \), \( f(O'_2) \), \( O'_3 \), \( f(O'_3) \), \( \ldots \) is a play played according to \( f \) and for all \( k \geq 0 \), \( O'_{k+1} \in \mathcal{O}(O_{2n_k+1}) \). We deduce that \( \cap_{n \geq 1} O'_n = \{x\} \) and as \( f \) is a winning strategy, we conclude that \( x \in W \). \[ \square \]

We now generalise the notion of \( \alpha \)-strategy to Banach--Mazur games on metric spaces.
Definition 6.6. Let $G = (X, W)$ be a Banach–Mazur game where $(X, d)$ is a metric space, $P$ a Borel probability measure on $X$ with full support (i.e. for each non-empty open set $O$, $P(O) > 0$) and $\alpha > 0$. An $\alpha$-strategy of Pl. 0 for $G$ is a strategy of Pl. 0 for the generalised Banach–Mazur game $(G, \Phi_U, \Phi_{\text{ball}}, W)$ where for all $O \in \mathcal{O}(X)$,

$$
\phi_\alpha(O) = \left\{ O' \in \mathcal{O}(O) : P(O' \mid O) \geq \alpha \right\}.
$$

In order to prove that the existence of a winning $\alpha$-strategy for some $\alpha > 0$ implies $P(W) = 1$ (as in the case of finite graphs), we need to suppose that $(X, d)$ is a separable space, i.e., there exists a dense countable set $D$ in $X$. We notice that in the case of a finite graph $G$, the space of infinite paths endowed with the metric given by (2.1) was also separable.

We start off by recalling some basic results of probability for separable metric spaces.

Lemma 6.7. (See [3].) Let $X$ be a metric space and $P$ a Borel probability measure on $X$. Then for all Borel set $E$ in $X$,

$$
P(E) = \inf\{ P(U) : U \supseteq E, U \text{ open} \}.
$$

Lemma 6.8. (See [11].) Let $X$ be a separable metric space and $P$ a Borel probability measure on $X$. Then for all $m \geq 1$, there are countably many open balls $(B_i)_{i \geq 1}$ such that $\bigcup_{i=1}^{\infty} B_i = X$ and for all $i \geq 1$, $\text{diam}(B_i) \leq \frac{1}{m}$ and $P(\partial B_i) = 0$ where $\partial B_i = B_i \setminus B_i$.

Corollary 6.9. Let $X$ be a separable metric space and $P$ a Borel probability measure on $X$. Then for all $m \geq 1$, there exists a countable set $(A_i)_{i \geq 1}$ of disjoint open subsets of $X$ with diameter less than $\frac{1}{m}$ such that

$$
P\left( \bigcup_{i=1}^{\infty} A_i \right) = 1.
$$

Proof. Let $(B_i)_{i \geq 1}$ be a sequence of open balls such that $\bigcup_{i=1}^{\infty} B_i = X$ and for all $i \geq 1$, $\text{diam}(B_i) \leq \frac{1}{m}$ and $P(\partial B_i) = 0$. If we let $A_1 = B_1$ and $A_i = B_i \setminus \bigcup_{j=1}^{i-1} B_j$, then the sequence $(A_i)$ is a sequence of disjoint open sets with diameter less than $\frac{1}{m}$ such that $P\left( \bigcup_{i=1}^{\infty} A_i \right) = 1$ since $\bigcup_{i=1}^{\infty} A_i \supseteq \left( \bigcup_{i=1}^{\infty} B_i \right) \setminus \left( \bigcup_{i=1}^{\infty} \partial B_i \right)$. □

Theorem 6.10. Let $G = (X, W)$ be a Banach–Mazur game on a separable complete metric space $(X, d)$ and $P$ a Borel probability measure with full support. If Pl. 0 has a winning $\alpha$-strategy for some $\alpha > 0$, then $P(W) = 1$.

Proof. Let $f$ be a winning $\alpha$-strategy. We first notice that if for all $n \geq 1$, $A_n$ is a family of open subsets of diameter less than $\frac{1}{n}$, then

$$
W \supset \left\{ x \in X : \# \{ n : x \in \bigcap_{A \in A_n} f(A) \} = \infty \right\}.
$$

Indeed, if $x \in \bigcup_{A \in A_n} f(A)$ for an infinity of indices $n$, then we can see $x$ as the result of a play of $(G, \Phi_U, \Phi_{\text{ball}}, W)$ played according to $f$ because for all $n \geq 1$, if $x \in f(A_n)$ for some $A_n \in A_n$, then we can find $m > n$ and $A_m \in A_m$ such that $B(x, 2/m) \subset f(A_n)$ and $x \in f(A_m)$. Therefore, since $\text{diam}(A_m) < 1/m$, we deduce that $A_m \subset f(A_n)$ and we can thus construct a sequence $(A_n)_{n \geq 1}$ such that $(A_n)$ is increasing, $x \in A_n \subset A_{n+1}$ and $A_{n+1} \subset f(A_n)$.

For all $n \geq 1$, we consider a countable set $A_n$ of disjoint open sets with diameter less than $\frac{1}{n}$ such that

$$
P\left( \bigcup_{A \in A_n} A \right) = 1 \quad \text{(Corollary 6.9)}.
$$

By Proposition 6.5, we can suppose that for all non-empty open sets $O \subset O', f(O') \subset f(O)$.

Let $E$ be a Borel subset of $X$ and $j \geq 1$. We therefore have for all open set $U \supseteq E$,

$$
P\left( \left\{ x \in X : x \in E \cap \bigcap_{A \in A_j} f(A)^c \right\} \right) = P\left( E \cap \bigcap_{A \in A_j} f(A)^c \right) \leq P\left( E \setminus \bigcup_{A \in A_j} f(A) \right)
\leq P(U) - \sum_{A \in A_j} P(f(A) \cap U)
\leq P(U) - \sum_{A \in A_j} P(f(A) \cap U)
\leq P(U) - \alpha \sum_{A \in A_j} P(A \cap U)
\leq (1 - \alpha) P(U)
$$

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and since \( P(E) = \inf \{ P(U) : U \supseteq E, U \text{ open} \} \) (Lemma 6.7), we get
\[
P \left( \bigcap_{A \in A_j} f(A)^c \right) \leq (1 - \alpha) P(E).
\]

We deduce from (6.2) with \( E = X \) that for all \( j \geq 1 \),
\[
P \left( \bigcap_{A \in A_j} f(A)^c \right) \leq (1 - \alpha)
\]
and by induction that for all \( k \leq m \),
\[
P \left( \bigcap_{j=k}^m \bigcap_{A \in A_j} f(A)^c \right) \leq (1 - \alpha)^{m+1-k}.
\]

We conclude that
\[
P \left( \{ x \in X : \exists k \geq 1, \forall j \geq k, x \notin \bigcup_{A \in A_j} f(A) \} \right) = 0.
\]

The result then follows from (6.1). \( \square \)

On the other hand, if \( W \) is a countable intersection of open sets of probability 1, we can easily prove that Pl. 0 has a winning \( \alpha \)-strategy for all \( 0 < \alpha < 1 \).

**Theorem 6.11.** Let \( G = (X, W) \) be a Banach–Mazur game on a complete metric space \( X \) where \( W \) is a countable intersection of open sets and \( P \) a Borel probability measure on \( X \) with full support. If \( P(W) = 1 \), then Pl. 0 has a winning \( \alpha \)-strategy for all \( 0 < \alpha < 1 \).

**Proof.** Let \( 0 < \alpha < 1 \) and \( W = \bigcap_{n=1}^\infty W_n \) where \( W_n \) is an open set of probability 1. We can define a winning \( \alpha \)-strategy \( f \) of Pl. 0 as follows: if \( O \subset \bigcap_{k=1}^n W_k \) and \( O \not\subset W_n \), we let \( f_\alpha(O) = O \cap W_n \) (the function \( f \) is well an \( \alpha \)-strategy since \( P(W_n) = 1 \)). \( \square \)

In summary, we have the following result for separable complete metric space.

**Theorem 6.12.** Let \( G = (X, W) \) be a Banach–Mazur game on a separable complete metric space \( X \) where \( W \) is a countable intersection of open sets and \( P \) a Borel probability measure with full support. Then the following assertions are equivalent:

1. \( P(W) = 1 \),
2. Pl. 0 has a winning \( \alpha \)-strategy for some \( \alpha > 0 \),
3. Pl. 0 has a winning \( \alpha \)-strategy for all \( 0 < \alpha < 1 \).

We finish this section by proving that the generalised Banach–Mazur game \( G = (G, \phi_\alpha, \phi_{ball}, W) \) is determined when \( W \) is a countable intersection of open sets and \( X \) is a separable complete metric space.

**Theorem 6.13.** Let \( G_\alpha \) be the generalised Banach–Mazur game given by \( (X, \phi_\alpha, \phi_{ball}, W) \) where \( X \) is a separable complete metric space, \( W \) is a countable intersection of open sets and \( P \) is a Borel probability measure with full support. Then the following assertions are equivalent:

1. \( P(W) < 1 \),
2. Pl. 1 has a winning strategy for \( G_\alpha \) for some \( \alpha > 0 \),
3. Pl. 1 has a winning strategy for \( G_\alpha \) for all \( 0 < \alpha < 1 \).

**Proof.** We deduce from Theorem 6.11 that 2. \( \Rightarrow \) 1. because \( G_\alpha \) is a zero-sum game, and 3. \( \Rightarrow \) 2. is obvious.

1. \( \Rightarrow \) 3. Let \( W = \bigcap_{n=1}^\infty W_n \) with \( P(W) < 1 \) where each \( W_n \) is an open set. We know that there exists \( n \geq 1 \) such that \( P(W_n) < 1 \). We can thus assume without loss of generality that \( W \) is an open set since it is sufficient to prove that Pl. 1 has a winning strategy for \( (X, \phi_\alpha, \phi_{ball}, W) \).

Let \( 0 < \alpha < 1 \). We first remark that all plays \( (O_n) \) of \( G_\alpha \) are winning for Pl. 1 if for all \( n \geq 1 \), \( P(W|O_n) < 1 \). Indeed, by Proposition 6.5, we know that \( \bigcap_{n \geq 1} O_n = \{ x \} \) for some \( x \in X \). Therefore, as \( W \) is open and \( \text{diam}(O_n) \) tends to 0, \( x \in W \) if and only if \( O_n \subset W \) for some \( n \geq 1 \).
It remains to prove that for all non-empty open set \( O \) such that \( P(W|O) < 1 \), there exists a non-empty open set \( O_1 \in \phi_{ball}(O) \) such that \( P(W|O_1) < \alpha \), and to prove that if \( Pl \) 1 plays such an open set \( O_1 \) and \( Pl \) 0 plays an open set \( O_2 \subset O_1 \) such that \( P(O_2|O_1) \geq \alpha \) then \( P(W|O_2) < 1 \).

Let \( O \) be an open set such that \( P(W|O) < 1 \) and \( r = \text{diam}(O) \). Let \((x_n)\) be a dense sequence in \( O \). For all \( n \geq 1 \), there exists \( r_n < \frac{1}{n} \) such that \( O = \bigcup_{n=1}^{\infty} B(x_n, r_n) \). Therefore, there exists \( n \geq 1 \) such that \( P(W^c \cap B(x_n, r_n)) > 0 \) and there exists \( r'_n < r_n \) such that

\[
C := P(W^c \cap B(x_n, r'_n)) > 0.
\]

Let \( \varepsilon < C \alpha \) and let \( U \supseteq W^c \) be an open set such that \( P(U) \leq P(W^c) + \varepsilon \). We let

\[
O_1 = U \cap B(x_n, r'_n).
\]

Thereby, we get that \( O_1 \) is an open set, \( O_1 \subset B(x_n, r'_n) \subset B(x_n, r_n) \subset O \) and

\[
2\text{diam}(O_1) \leq 2\text{diam}(B(x_n, r_n)) \leq 4r_n \leq r = \text{diam}(O).
\]

We deduce that \( O_1 \in \phi_{ball}(O) \). Moreover, since \( P(U \cap W) = P(U) - P(W^c) \leq \varepsilon \), we get

\[
P(W|O_1) = \frac{P(W \cup U \cap B(x_n, r'_n))}{P(U \cap B(x_n, r'_n))} \leq \frac{P(W \cup U)}{C} \leq \frac{\varepsilon}{C} < \alpha.
\]

Now, if we consider an open set \( O_2 \subset O_1 \) such that \( P(O_2|O_1) \geq \alpha \), then \( P(O_2) \geq \alpha P(O_1) \) and

\[
P(O_1 \cap O_2) = P(O_1) - P(O_2) \leq (1 - \alpha)P(O_1).
\]

Moreover, since \( P(W|O_1) < \alpha \), we have \( P(W^c \cap O_1) > (1 - \alpha)P(O_1) \). We also remark that \( P(W|O_2) < 1 \) if and only if \( P(W^c \cap O_2) > 0 \). Therefore, we can conclude because

\[
P(W^c \cap O_2) = P(W^c \cap O_1) - P(W^c \cap O_1 \cap O_2)
\]

\[
> (1 - \alpha)P(O_1) - P(O_1 \cap O_2)
\]

\[
> (1 - \alpha)P(O_1) - (1 - \alpha)P(O_1) = 0.
\]

\[\Box\]

**Corollary 6.14.** Let \( 0 < \alpha < 1 \). The generalised Banach–Mazur game \( G_\alpha = (X, \phi_\alpha, \phi_{ball}, W) \) is determined when \( X \) is a separable complete metric space, \( W \) is a countable intersection of open sets and \( P \) is a Borel probability measure with full support. More precisely:

1. \( Pl \) 0 has a winning strategy if and only if \( P(W) = 1 \).
2. \( Pl \) 1 has a winning strategy if and only if \( P(W) < 1 \).

**References**


