# Compositional Design of Stochastic Timed Automata*

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**Abstract.** In this paper, we study the model of stochastic timed automata and we target the definition of adequate composition operators that will allow a compositional approach to the design of stochastic systems with hard real-time constraints. This paper achieves the first step towards that goal. Firstly, we define a parallel composition operator that (we prove) corresponds to the interleaving semantics for that model; we give conditions over probability distributions, which ensure that the operator is well-defined; and we exhibit problematic behaviours when this condition is not satisfied. We furthermore identify a large and natural subclass which is closed under parallel composition. Secondly, we define a bisimulation notion which naturally extends that for continuous-time Markov chains. Finally, we importantly show that the defined bisimulation is a congruence w.r.t. the parallel composition, which is an expected property for a proper modular approach to system design.

## 1 Introduction

Compositional design and compositional verification are two crucial aspects of the development of computerised systems for which correctness needs to be guaranteed or quantified. It is indeed convenient and natural to model separately each component of a system and model their interaction, and it is easier and probably less error-prone than to model at once the complete system.

In the last twenty years a huge effort has been made to design expressive models, with the aim to faithfully represent computerised systems. This is for instance the case of systems with real-time constraints for which the model of timed automata [1,2] is successfully used. Many applications like communication protocols require models integrating both real-time constraints and randomised aspects (see e.g. [24]), which requires the development of specific models. Recently, a model of stochastic timed automata (STA) has been proposed as a natural extension of timed automata with stochastic delays and stochastic edge choices (see [8] for a survey of the results so far concerning this model). Advantages of the STA model are twofold: (i) it is based on the well-understood and powerful

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model of timed automata, allowing to express hard real-time constraints like deadlines (unlike for the widely used model of *continuous-time Markov chains* (CTMCs in short)); (ii) it enjoys nice decidability properties (see [8,9]). On the other hand, there is no obvious way of designing in a compositional manner a complex system using this model.

In this paper we are inspired by the approach of [23], and we target the definition of (parallel) composition operators allowing for a component-based modelling framework for STA. This paper achieves the first steps towards that goal:

1. We define a parallel composition operator that (we prove) corresponds to the interleaving semantics for that model; we give conditions over families of distributions over delays, which ensure that the operator is well-defined; we exhibit problematic behaviours when this condition is not satisfied. We furthermore identify a class of such well-behaving STA that is closed under parallel composition. Note that this class of well-behaving systems encompasses the class of CTMCs.
2. We define a bisimulation notion which naturally extends that for CTMCs [5,6,16], and we importantly show that the bisimulation is a congruence w.r.t. parallel composition; this is an expected property for a proper modular approach to system design.

The next step will be to extend the current composition operator with some synchronisation between components. For CTMCs, this has required much effort over the years to come up with a satisfactory solution, yielding for instance the model of *interactive Markov chains* (IMCs) [20,21]. We believe we will benefit a lot from this solution and plan to follow a similar approach for STA; we leave it as further work (the current work focuses on races between components and establishes all useful properties at the level of STA).

*Related works* We do not list all works concerned with the verification of stochastic real-time systems, but will focus on those interested in compositional design. The first natural related work is that on interactive Markov chains (IMCs in short) [20,21], which extend CTMCs with interaction, and for which compositional verification methods have been investigated [12,22]. However in this model, only soft real-time constraints can be evaluated (that is, they may not be always satisfied by the system, but their likelihood is then quantified), and the model cannot evolve differently, depending on constraints over clocks. Our ultimate goal is to extend the elegant approach of IMCs to a model based on timed automata.

Other related approaches are based on process algebras (note that originally IMCs presented as a process algebra as well [20]). There have been several proposals, among which the IGSMP calculus [11], whose semantics is given as generalised semi-Markov processes (GSMPs); and the stochastic process algebra $\spadesuit$ [14,15], whose semantics is given as $\spadesuit$-stochastic timed automata (we write $\spadesuit$-STA). Our model very much compares to the latter, so we will briefly describe it. In such a system, when a clock variable is activated, it is sampled according to a predefined distribution, and then it acts as a countdown timer: when time elapses, the clock variables decrease down to 0. Transitions can be fired once all clocks specified on the transition have reached value 0. First notice that both STA and ♤-STA allow to express hard real-time constraints, e.g. strict deadlines to be satisfied by the system (which is not the case of CTMCs or IMCs). Then the ♤-STA model is at the basis of several modelling languages like Modest [10] and comes with several notions of bisimulations with nice congruence properties, and with a complete equational theory. It is interesting to mention as well that ♤-STA allow for infinitely many states and clock variables, whereas STA do not (they have been defined on top of timed automata, with desirable decidability properties in mind). Similarly to ♤-STA, STA extend (finite-state and finite-variable) GSMPs,[1] but for different reasons: ♤-STA allows for fixed-delay events and non-determinism, whereas STA allows for more intricate timing constraints and branchings.[2] Finally, it is worth mentioning the modelling language Modest [10], whose semantics is given as a very general notion of stochastic timed automata (we call them Modest-STA), which comes with an interesting tool suite [18,19], and which encompasses all the models we have mentioned. STA in general, and the subclass that is closed under parallel composition while enjoying decidability properties, can be viewed as a fragment of Modest-STA (see Appendix A.1 for more details).

Full proofs are given in a separate appendix.

## 2 Stochastic Timed Automata

In this section, we recall the notion of *timed automaton* [2], and that of *stochastic timed automaton* [8]. Let $X = \{x_1, \ldots, x_n\}$ be a finite set of real-valued variables called *clocks*. A *clock valuation* over $X$ is a mapping $\nu : X \to \mathbb{R}_+$ where $\mathbb{R}_+$ is the set of nonnegative real numbers. We write $\mathbb{R}_+^X$ for the set of clock valuations over $X$. If $\nu \in \mathbb{R}_+^X$, we write $\nu_i$ for $\nu(x_i)$ and we then denote $\nu$ by $(\nu_1, \ldots, \nu_n)$. If $\tau \in \mathbb{R}_+$, we write $\nu + \tau$ for the clock valuation defined by $(\nu_1 + \tau, \ldots, \nu_n + \tau)$. If $Y \in 2^X$ (the power set of $X$), $[Y \leftarrow 0]\nu$ is the valuation that assigns to $x$, 0 if $x \in Y$ and $\nu(x)$ otherwise. A *guard*[3] over $X$ is a finite conjunction of expressions of the form $x_i \sim c$ where $c \in \mathbb{N}$ and $\sim \in \{<, >\}$. We denote by $\mathcal{G}(X)$ the set of guards over $X$. We write $\nu \models g$ if $\nu$ satisfies $g$, which is defined in a natural way.

**Definition 1.** *A timed automaton (TA in short) is a tuple $\mathcal{A} = (L, L_0, X, E, \mathsf{AP}, \mathcal{L})$ where: (i) $L$ is a finite set of locations, (ii) $L_0 \subseteq L$ is a set of initial locations, (iii) $X$ is a finite set of clocks, (iv) $E \subseteq L \times \mathcal{G}(X) \times 2^X \times L$ is a finite set of edges, (v) $\mathsf{AP}$ is a set of atomic propositions and (vi) $\mathcal{L} : L \to 2^{\mathsf{AP}}$ is a labelling function.*

The semantics of a TA is a labelled timed transition system $T_{\mathcal{A}} = (Q, Q_0, \mathbb{R}_+ \times E, \to, \mathsf{AP}, \mathcal{L})$ where $Q = L \times \mathbb{R}_+^X$ is the set of states, $Q_0 = L_0 \times \mathbf{0}_X$ is the set

[1] This can be seen using the residual-time semantics given in [17,13].

[2] Somehow, the clock behaviour in GSMPs and in ♤-STA is that of countdown timers (which can be seen as event-predicting clocks of [3]), which is not as rich as general clocks in standard timed automata.

[3] We restrict to open guards for technical reasons due to stochastic aspects.

of initial states (valuation $\mathbf{0}_X$ assigns 0 to each clock), $\mathcal{L} : Q \to 2^{\mathsf{AP}}$ labels each state $q = (l, \nu) \in Q$ by $\mathcal{L}(l)$ and $\to \subseteq Q \times (\mathbb{R}_+ \times E) \times Q$ is the transition relation defined as follows: if $e = (l, g, Y, l') \in E$ and $\tau \in \mathbb{R}_+$, then we have $(l, \nu) \xrightarrow{\tau, e} (l', \nu')$ if $(\nu + \tau) \models g$ and $\nu' = [Y \leftarrow 0](\nu + \tau)$. If $q = (\ell, \nu)$, for every $\tau \geq 0$, $q + \tau$ denotes $(\ell, \nu + \tau)$. A *finite* (resp. *infinite*) *run* $\rho$ is a finite (resp. infinite) sequence $\rho = q_1 \xrightarrow{\tau_1, e_1} q_2 \xrightarrow{\tau_2, e_2} \ldots$. Given $q \in Q$, we write $\mathrm{Runs}(\mathcal{A}, q)$ for the set of infinite runs in $\mathcal{A}$ from $q$. Given $q \in Q$ and $e \in E$ we define $I(q, e) = \{\tau \in \mathbb{R}_+ \mid \exists q' \in Q \text{ s.t. } q \xrightarrow{\tau, e} q'\}$ and $I(q) = \bigcup_{e \in E} I(q, e)$.

We now define the notion of *stochastic timed automaton* [8], by equipping every state of a TA with probability measures over both delays and edges.

**Definition 2.** *A stochastic timed automaton (STA in short) is a tuple* $\mathcal{A} = (L, L_0, X, E, AP, \mathcal{L}, (\mu_q, p_q)_{q \in L \times \mathbb{R}_+^X})$ *where* $(L, L_0, X, E, AP, \mathcal{L})$ *is a timed automaton and for every* $q = (l, \nu) \in L \times \mathbb{R}_+^X$*,*

- *(i)* $\mu_q$ *is a probability distribution over* $I(q)$ *and* $p_q$ *is a probability distribution over* $E$ *such that for each* $e = (l, g, Y, l') \in E$*,* $p_q(e) > 0$ *iff* $\nu \models g$*,*
- *(ii)* $\mu_q$ *is equivalent to the restriction of the Lebesgue measure on* $I(q)$*,*[4] *and*
- *(iii) for each edge* $e$*, the function* $p_{q+\bullet}(e) : \mathbb{R}_+ \to [0, 1]$ *that assigns to each* $t \geq 0$ *the value* $p_{q+t}(e)$*, is measurable.*

We fix $\mathcal{A}$ a STA, with the notations of the definition. We let $Q = L \times \mathbb{R}_+^X$ be the set of states of $\mathcal{A}$, and pick $q \in Q$. We aim at defining a probability distribution $\mathbb{P}_\mathcal{A}$ over $\mathrm{Runs}(\mathcal{A}, q)$. Let $e_1, \ldots, e_k$ be edges of $\mathcal{A}$, and $\mathcal{C} \subseteq \mathbb{R}_+^k$ be a Borel set. The *(constrained) symbolic path* starting from $q$ and determined by $e_1, \ldots, e_k$ and $\mathcal{C}$ is the following set of finite runs: $\pi_\mathcal{C}(q, e_1, \ldots, e_k) = \{\rho = q \xrightarrow{\tau_1, e_1} q_1 \cdots \xrightarrow{\tau_k, e_k} q_k \mid (\tau_1, \ldots, \tau_k) \in \mathcal{C}\}$. Given a symbolic path $\pi$, we define the cylinder generated by $\pi$ as the subset $\mathrm{Cyl}(\pi)$ of $\mathrm{Runs}(\mathcal{A}, q)$ containing all runs $\rho$ with a prefix $\rho'$ in $\pi$.

We inductively define a measure over the set of symbolic paths as follows:

$$\mathbb{P}_\mathcal{A}(\pi_\mathcal{C}(q, e_1, \ldots, e_k)) = \int_{t_1 \in I(q, e_1)} p_{q+t_1}(e_1)\, \mathbb{P}_\mathcal{A}(\pi_{\mathcal{C}_{[\tau_1/t_1]}}(q_{t_1}, e_2, \ldots, e_k))\, \mathrm{d}\mu_q(t_1),$$

where for every $t_1 \geq 0$, $q_{t_1}$ is such that $q \xrightarrow{t_1, e_1} q_{t_1}$ and $\mathcal{C}_{[\tau_1/t_1]}$ replaces variable $\tau_1$ by $t_1$ in $\mathcal{C}$; we initialise with $\mathbb{P}_\mathcal{A}(\pi(q)) = 1$. The formula for $\mathbb{P}_\mathcal{A}$ relies on the fact that the probability of taking transition $e_1$ at time $t_1$ coincides with the probability of waiting $t_1$ time units and then choosing $e_1$ among the enabled transitions, i.e. $p_{q+t_1}(e_1)\, \mathrm{d}\mu_q(t_1)$. Now, one can extend $\mathbb{P}_\mathcal{A}$ to the cylinders by $\mathbb{P}_\mathcal{A}(\mathrm{Cyl}(\pi)) = \mathbb{P}_\mathcal{A}(\pi)$, where $\pi$ is a symbolic path. Using some extension theorem as Carathéodory's theorem, we can extend $\mathbb{P}_\mathcal{A}$ in a unique way to the $\sigma$-algebra generated by the cylinders starting in $q$, which we denote $\Omega_\mathcal{A}^q$.

**Proposition 1 ([8]).** *Let* $\mathcal{A} = (L, L_0, X, E, \mathsf{AP}, \mathcal{L}, (\mu_q, p_q)_{q \in L \times \mathbb{R}_+^X})$ *be a STA. For every state* $q \in Q$*,* $\mathbb{P}_\mathcal{A}$ *is a probability measure over* $(\mathrm{Runs}(\mathcal{A}, q), \Omega_\mathcal{A}^q)$.

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[4] Two measures $\mu$ and $\nu$ on the same measurable space are equivalent whenever for every measurable set $A$, $\mu(A) > 0$ iff $\nu(A) > 0$.

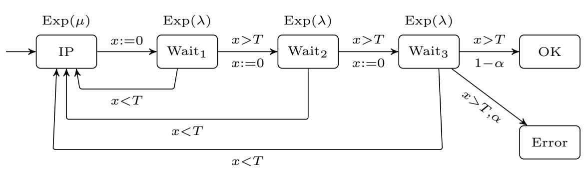


**Fig. 1.** The IPv4 Zeroconf STA.

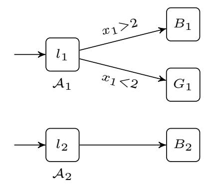


**Fig. 2.** $\mathcal{A}_1 \notin$ CSTA.

*Remark 1.* Among others, the set of Zeno runs is measurable in $\Omega^q_{\mathcal{A}}$;[5] writing $\mathcal{C}_{M,k}$ for $\{(\tau_1, \ldots, \tau_k) \in \mathbb{R}^k_+ \mid \tau_1 + \ldots + \tau_k \leq M\}$ it is indeed expressible as follows:

$$\bigcup_{M \in \mathbb{N}} \bigcap_{k \in \mathbb{N}_0} \bigcup_{(e_1, \ldots, e_k) \in E^k} \mathrm{Cyl}(\pi_{\mathcal{C}_{M,k}}(q, e_1, \ldots, e_k)).$$

*Remark 2.* A CTMC can be viewed as a STA with trivial guards on transitions and exponential distributions over delays.

We now give an example of STA. A further example of a G/G/1/$k$-queue is given in Appendix B.

*Example 1.* We model the IPv4 Zeroconf protocol using STA as done in [8] (see Figure 1). This protocol aims at configuring IP addresses in a local network of appliances. When a new appliance is plugged, it selects an IP address at random, and broadcasts several probe messages to the network to know whether this address is already used or not. If it receives in a bounded delay an answer from the network informing that the IP is already used, then a new IP address is chosen. It may be the case that messages get lost, in which case there is an error. In [7], a simple model for the IPv4 Zeroconf protocol is given as a discrete-time Markov chain, which abstracts away timing constraints. In Figure 1, we model the protocol as a STA with a single clock $x$, and exponential distributions (of parameters $\mu$ and $\lambda$) and this allows us to explicitly express the delay bound.

*Discussion on the model.* STA have been defined and studied in a series of papers from 2007, with a complete journal version published as [8]. They can be used for modelling systems with stochastic aspects and real-time constraints (they are based on the standard model of timed automata [2] and extend the model of CTMCs) and are amenable to automatic verification. The class of *almost-surely fair STA*[6] is of particular interest. Indeed:

**Theorem 1 ([8]).** *The almost-sure model-checking problem is decidable for the class of almost-surely fair STA, with regards to $\omega$-regular properties or properties given as deterministic timed automata.*

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[5] We recall that a run $\rho = q \xrightarrow{\tau_1, e_1} q_1 \xrightarrow{\tau_2, e_2} \ldots$ is *Zeno* if $\sum_{i \geq 1} \tau_i < +\infty$.

[6] A STA is said almost-surely fair whenever $\mathbb{P}_{\mathcal{A}}(\mathsf{fair}) = 1$, where a run is fair if and only if (roughly speaking) any edge enabled infinitely often is taken infinitely often.

There exists surprisingly simple examples of STA which are not almost surely fair (see for example [8, Figure 9]), but large classes of STA have been identified in [8], that are almost-surely fair (they include single-clock STA and (almost-)reactive STA). Deciding whether a STA is almost-sure fair is an open problem

The approach adopted so far for modelling and verifying is monolithic. We target modular design of STA and describe a class of STA in which composition can safely be applied.

## 3 Parallel Composition of Stochastic Timed Automata

Compositional design is desirable for building computerised systems. Inspired by the approach of [23], we first define a parallel composition operator for STA, which corresponds to an interleaving semantics. This operator involves complex behaviours that are due to races between components. We therefore give conditions under which STA can be safely composed.

*Remark 3.* As already mentioned earlier, we focus here on an interleaving parallel composition operator between STA, and study the races between components. Extension to a parallel composition operator with some synchronisation is part of our future work, and we plan to adopt the idea of interactive Markov chains [20,21], which extend CTMCs with interactive actions, for the purpose of synchronisation.

### 3.1 Definition of the parallel composition

We consider two STA $\mathcal{A}_i = (L_i, L_0^{(i)}, X_i, E_i, \mathsf{AP}_i, \mathcal{L}_i, (\mu_q^{(i)}, p_q^{(i)})_{q \in L_i \times \mathbb{R}_+^{X_i}})$ for $i = 1, 2$ with $X_1 \cap X_2 = \emptyset$, and we first recall the standard (interleaving) parallel composition for the underlying TA. It is the TA $(L, L_0, X, E, \mathsf{AP}, \mathcal{L})$ where $L = L_1 \times L_2$, $L_0 = L_0^{(1)} \times L_0^{(2)}$, $X = X_1 \cup X_2$, $\mathsf{AP} = \mathsf{AP}_1 \cup \mathsf{AP}_2$, $\mathcal{L} : L \to 2^{\mathsf{AP}}$ is such that $\mathcal{L}((l_1, l_2)) = \mathcal{L}_1(l_1) \cup \mathcal{L}_2(l_2)$ and where $E = E_{1,\bullet} \cup E_{\bullet,2}$ with $E_{1,\bullet} = \{((l_1, l_2), g, Y, (l_1', l_2)) \mid (l_1, g, Y, l_1') \in E_1,\ l_2 \in L_2\}$.

Back to the STA, the parallel composition $\mathcal{A}_1 \parallel \mathcal{A}_2$ has as underlying TA the one above; it remains to equip each state $q = (q_1, q_2) \in Q_1 \times Q_2$ with probability distributions over both delays and edges, with the following constraints:

- distributions over delays from state $(q_1, q_2)$ should reflect a *race* between the two components $\mathcal{A}_1$ and $\mathcal{A}_2$ from respectively states $q_1$ and $q_2$;
- distributions over edges should be state-based (or memoryless), that is, should not depend on how long has been waited before taking that edge, or which other actions have been done meanwhile by other components;
- globally, the product-automaton should correspond to the interleaving of $\mathcal{A}_1$ and $\mathcal{A}_2$, which we express as follows: given a property $\varphi_1$ that only concerns $\mathcal{A}_1$ and a property $\varphi_2$ that only concerns $\mathcal{A}_2$, $\mathbb{P}_{\mathcal{A}_1 \parallel \mathcal{A}_2}(\varphi_1 \wedge \varphi_2) = \mathbb{P}_{\mathcal{A}_1}(\varphi_1) \cdot \mathbb{P}_{\mathcal{A}_2}(\varphi_2)$.

Example 2 will illustrate the intricacy of getting these conditions satisfied.

Let $\mathcal{A}$ be a STA and let $q = (l, \nu) \in Q$ be a state of $\mathcal{A}$. We write $f_q$ for the density function of $\mu_q$ w.r.t. the Lebesgue measure. We write $F_q$ for the cumulative function associated to $f_q$.

We now define a first class of STA, called CSTA, which is suitable to define a parallel composition. We say that a STA $\mathcal{A}$ is in CSTA if:

(A) for every state $q$ of $\mathcal{A}$, the density function associated with $\mu_q$, denoted by $f_q$, is continuous everywhere on $\mathbb{R}_+$ except in a finite number of points, and
(B) the family of probability distributions $(\mu_q)_{q \in Q}$ is *weakly-memoryless*, i.e. for every $t, t' \geq 0$, $\mathbb{P}_{\mathcal{A}}(\mathbb{X}_q \geq t + t' \mid \mathbb{X}_q \geq t) = \mathbb{P}_{\mathcal{A}}(\mathbb{X}_{q+t} \geq t')$, where $\mathbb{X}_q$ (resp. $\mathbb{X}_{q+t}$) is a random variable with density function $f_q$ (resp. $f_{q+t}$).

This second condition is a consistency condition between states which belong to the same 'time-elapsing fiber', that is, sets of the form $F = \{q + t \mid t \in \mathbb{R} \text{ and } q + t \in Q\}$. Indeed, $\mathbb{X}_q$ (resp. $\mathbb{X}_{q+t}$) represents the delay after which we leave state $q$ (resp. $q+t$) via an edge. Hence if $q_0$ is the minimal (for time-elapsing) element of $F$, then for every $q = q_0 + t \in F$, the law of $\mathbb{X}_q$ has to be equal to the law of $\mathbb{X}_{q_0}$ conditioned by the fact that $t$ time units have already passed. The distribution in $q_0$ can be taken arbitrary (satisfying condition (A)), and distributions for $q \in F$ can then be inferred.

Condition (B) can equivalently be written as: for every $t, t' \geq 0$,

$$f_q(t + t') = (1 - F_q(t)) f_{q+t}(t') \tag{1}$$

*Remark 4.* Let $q_0$ be an initial element of a fiber, we can check that for instance,

- if $I(q_0)$ is a bounded subset of $\mathbb{R}_+$ and if $\mu_{q_0}$ is a uniform distribution over $I(q_0)$, then for every $t \in \mathbb{R}_+$, (B) imposes that $\mu_{q_0+t}$ is also uniform over $I(q_0 + t)$;
- similarly, if $I(q_0) = \mathbb{R}_+$, and if $\mu_{q_0}$ is an exponential distribution with parameter $\lambda$ (denoted $\text{Exp}(\lambda)$), then for every $t \in \mathbb{R}_+$, (B) imposes that $\mu_{q_0+t}$ is also an $\text{Exp}(\lambda)$-distribution. This corresponds to the classical memoryless property assumed in CTMCs.

We can now explain how to build the probability distributions associated with a state $q = (q_1, q_2)$ of $\mathcal{A}_1 \parallel \mathcal{A}_2$. Since we leave state $q = (q_1, q_2)$ as soon as we leave $q_1$ or $q_2$, we naturally define the distribution over the delays from $q$ as the minimum of the distributions over delays from $q_1$ and $q_2$. Under hypothesis (A) for the distributions from $q_1$ and $q_2$, one can show that the density function $f_q$ for the minimum satisfies $f_q(t) = f_{q_1}(t)(1 - F_{q_2}(t)) + f_{q_2}(t)(1 - F_{q_1}(t))$ almost-surely for every $t \geq 0$ (w.r.t. the Lebesgue measure).

In order to define the probability distribution $p_q$ over the enabled edges in $q$, one could consider that from state $q$, both systems $\mathcal{A}_1$ and $\mathcal{A}_2$ are in a race to win the next edge, i.e. $\mathcal{A}_1$ wins the race if the first edge taken from $q$ is in $E_1$. Hence, given $t \in I(q)$, and an edge $e \in E_1$ enabled in $q + t$, one would like that $p_{q+t}(e) = w_q^1(t) p_{q_1+t}(e)$ where $w_q^1(t)$ is the probability that, starting from

$q$, $\mathcal{A}_1$ wins the race knowing that it was won after a delay of $t$ time units. This can be formalized, and under hypothesis (A) for $f_{q_1}$ and $f_{q_2}$, we can show that if $f_q(t) \neq 0$, then $w_q^1(t) = \frac{f_{q_1}(t)(1-F_{q_2}(t))}{f_q(t)}$ almost-surely.

**Definition 3.** *Let* $\mathcal{A}_i = (L_i, L_0^{(i)}, X_i, E_i, AP_i, \mathcal{L}_i, (\mu_q^{(i)}, p_q^{(i)})_{q \in L_i \times \mathbb{R}_+^{X_i}})$ *for* $i = 1, 2$ *be two STA. We say that* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *are composable if* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *are in* CSTA *and* $X_1 \cap X_2 = \emptyset$*.In that case, we define the parallel composition of* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *as the STA* $\mathcal{A}_1 \parallel \mathcal{A}_2 = (L, L_0, X, E, AP, \mathcal{L}, (\mu_q, p_q)_{q \in L \times \mathbb{R}_+^X})$*, where for any state* $q = (q_1, q_2)$ *of* $\mathcal{A}_1 \parallel \mathcal{A}_2$*,*

*(i)* $(L, L_0, X, E, AP, \mathcal{L})$ *is the composition of the underlying TA* $\mathcal{A}_1$ *and* $\mathcal{A}_2$*,*
*(ii)* $\mu_q$ *is defined by its density function* $f_q = f_{q_1}(1 - F_{q_2}) + f_{q_2}(1 - F_{q_1})$*, and*
*(iii) for any* $t \in I(q)$*,* $p_{q+t}$ *is defined as follows:*

$$p_{q+t}(e) = \mathbb{1}_{E_1}(e) w_q^1(t) p_{q_1+t}(e) + \mathbb{1}_{E_2}(e) w_q^2(t) p_{q_2+t}(e)$$

*for every* $e \in E$*, where* $w_q^i = \frac{f_{q_i}}{f_q}(1 - F_{q_{3-i}})$ *on* $I(q)$*, for* $i = 1, 2$*.*

### 3.2 Properties of the parallel composition

We are now ready to prove that this parallel composition operator satisfies all the expected properties. We assume the notations of Definition 3. First:

**Lemma 1.** *The distributions* $\mu_q$ *and* $p_q$ *are well-defined, and the STA* $\mathcal{A}_1 \parallel \mathcal{A}_2$ *belongs to the class* CSTA*.*

We now give an example of a family of probability measures that do not satisfy hypothesis (B), which yields undesirable properties in the parallel composition.

*Example 2 (Counter-example for condition (B)).* We consider the single-clock STA $\mathcal{A}_1$ depicted in Figure 2 (page 5). We assume $\mu_{q_1}$ is an exponential distribution of parameter $\lambda_1$ (resp. $\lambda_1'$) if $q_1 = (l_1, \nu_1)$ with $\nu_1 < 1$ (resp. $\nu_1 \geq 1$), and with $\lambda_1 \neq \lambda_1'$. Then for each $\nu_1 \in [0, 1[$, $\mu_{q_1}$ does not satisfy hypothesis (B). We then compose $\mathcal{A}_1$ with the STA $\mathcal{A}_2$. Each state $q_2 = (l_2, \nu_2)$ is equipped with an exponential distribution of parameter $\lambda_2 = \lambda_1'$ over the delays. It can be shown that the probability to reach $B_1$ in $\mathcal{A}_1$ corresponds to the probability to reach $(B_1, B_2)$ in $\mathcal{A}_1 \parallel \mathcal{A}_2$ iff $\ln(\lambda_1) - \ln(\lambda_2) = \lambda_1 - \lambda_2$, which is not true in general.

*Example 3.* In order to illustrate the notion of composition, we composed two independent copies of the STA modelling the IPv4 Zeroconf protocol (see Example 1). Part of the composed STA is depicted in Figure 3.

It remains to identify when the parallel composition really coincides with an interleaving semantics. This is in general not true, as already shown in Example 2 (which does not satisfy Condition (B)), and witnessed further by Example 4 below (which satisfies both conditions (A) and (B)).

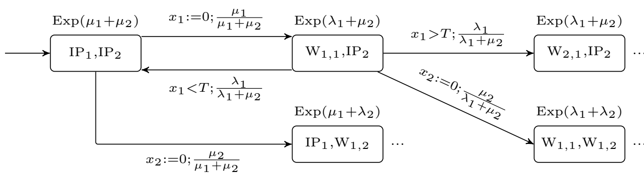


**Fig. 3.** The product of two STA modelling the IPv4 Zeroconf

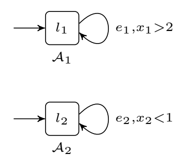


**Fig. 4.** $\mathcal{A}_2$ is Zeno

*Example 4.* We consider the STA $\mathcal{A}_1$ and $\mathcal{A}_2$ of Figure 4, equipped resp. with an Exp($\lambda$)-distribution and a uniform distribution. Let $q = (q_1, q_2)$ be a state of $\mathcal{A}_1 \parallel \mathcal{A}_2$, with $q_i = (l_i, 0)$. One can easily check that $\mathbb{P}_{\mathcal{A}_1 \parallel \mathcal{A}_2}(q \to^* \xrightarrow{e_1}) = 0$ while $\mathbb{P}_{\mathcal{A}_1}(\text{Cyl}(\pi(q_1, e_1))) = 1$ which contradicts the independence property we expect. One can notice that $\mathcal{A}_2$ is Zeno with probability 1.

Hence we define a subclass CSTA$^*$ of CSTA; $\mathcal{A} \in$ CSTA will be in CSTA$^*$ if:

(C) $\mathcal{A}$ is almost-surely non-Zeno.

*Remark 5.* Hypothesis (C) is not too restrictive since Zeno runs can be seen as faulty behaviours (they perform infinitely many actions in a finite amount of time, which is not realistic). We will see that hypothesis (C) is sufficient (together with (A) and (B)) to show that the parallel composition really coincides with an interleaving semantics. Note that condition (C) can be decided in various subclasses of STA [8].

We give some more notations. Let $\mathcal{A}$ be a STA and let $\varphi$ be a property for $\mathcal{A}$. Given a state $q$, we say that $\varphi$ is measurable from $q$ if the set of runs starting from $q$ satisfying $\varphi$ is in $\Omega^q_{\mathcal{A}}$; we write this set $\{q \models \varphi\}$. Now let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two composable STA. For $i = 1, 2$, we write $\iota_i$ for the natural projection of $\text{Runs}(\mathcal{A}_1 \parallel \mathcal{A}_2, (q_1, q_2))$ onto $\text{Runs}(\mathcal{A}_i, q_i)$, and given a measurable property $\varphi_i$ in $\mathcal{A}_i$ from $q_i$, we write $\{(q_1, q_2) \models \widetilde{\varphi}_i\}$ for the set $\iota_i^{-1}(\{q_i \models \varphi_i\})$. The following theorem states that the defined parallel composition is indeed interleaving.

**Theorem 2.** *Let $\mathcal{A}_1, \mathcal{A}_2 \in$ CSTA$^*$ be composable. Then $\mathcal{A}_1 \parallel \mathcal{A}_2 \in$ CSTA$^*$. Moreover, for every state $q = (q_1, q_2)$ of $\mathcal{A}_1 \parallel \mathcal{A}_2$, for every properties $\varphi_1$ measurable in $\mathcal{A}_1$ from $q_1$ and $\varphi_2$ measurable in $\mathcal{A}_2$ from $q_2$, we have*

$$\mathbb{P}_{\mathcal{A}_1 \parallel \mathcal{A}_2}(\{q \models \widetilde{\varphi}_1\} \cap \{q \models \widetilde{\varphi}_2\}) = \mathbb{P}_{\mathcal{A}_1}(\{q_1 \models \varphi_1\}) \cdot \mathbb{P}_{\mathcal{A}_2}(\{q_2 \models \varphi_2\}). \quad (2)$$

*Proof (Sketch).* Given $\mathcal{A}_1$ and $\mathcal{A}_2$ in CSTA$^*$, thanks to Lemma 1, it suffices to prove that $\mathcal{A}_1 \parallel \mathcal{A}_2$ is almost-surely non-Zeno. This will be ensured by (2) and the fact that *non-Zenoness* is a measurable property.

The important first step to prove (2) consists in showing that, given an edge $e_1$ of $\mathcal{A}_1$, the probability in $\mathcal{A}_1 \parallel \mathcal{A}_2$ that $e_1$ is the first edge from $\mathcal{A}_1$ (with possibly edges from $\mathcal{A}_2$ taken before) performed from $q = (q_1, q_2)$ in a given set of delays $\Delta$ corresponds to the probability in $\mathcal{A}_1$ that $e_1$ is the first edge performed from $q_1$ in the same set of delays $\Delta$. In order to do so, hypothesis (B) is crucial. The rest of the proof is long and technical but does not contain major difficulties. □

## 4 Bisimulation and Congruence

In this section, we define a notion of bisimulation for STA which naturally extends that for CTMCs [4,6,16]. We importantly show that the defined bisimulation is a congruence w.r.t. parallel composition: this means that, in a complex system, a component can be replaced by an equivalent one without affecting the global behaviour of the system.

### 4.1 Bisimulation

To define a bisimulation relation between STA, we are inspired by the approach of [16], which considers continuous-time Markov processes (CTMPs) – CTMPs generalize CTMCs to general continuous state-spaces; this definition of bisimulation that is given for CTMPs can be adapted to our context (note however that STA cannot be seen as particular CTMPs).

We first define some notions. A subset $P \subseteq \mathbb{R}^n$ is a *polyhedral set* if it is defined by a (finite) boolean combination of constraints of the form $A_1 x \leq b_1$ or $A_2 x < b_2$, where $x = (x_1, \ldots, x_n)$ is a variable, $A_1 \in \mathbb{R}^{m_1 \times n}$, $b_1 \in \mathbb{R}^{m_1}$, $A_2 \in \mathbb{R}^{m_2 \times n}$ and $b_2 \in \mathbb{R}^{m_2}$.

Let $\mathcal{A}$ be a STA, $Q$ be its set of states, and $P(Q) = \{\cup_{l \in L}\{l\} \times C_l \mid \forall l \in L,\ C_l \text{ polyhedral set of } \mathbb{R}_+^n\}$ where $n$ is the number of clocks of $\mathcal{A}$. The set $P(Q)$ is a proper subset of the Borel $\sigma$-algebra over $L \times \mathbb{R}_+^n$, which is closed by projection (contrary to the Borel $\sigma$-algebra). We then define the *closure of* $\mathcal{R}$ w.r.t. polyhedral sets, and we write $\mathrm{pcl}(\mathcal{R})$ as the following set $\mathrm{pcl}(\mathcal{R}) = \{A \in P(Q) \mid (a \in A \wedge a\mathcal{R}b) \Rightarrow b \in A\}$. One can notice that $\mathrm{pcl}(\mathcal{R})$ corresponds to the set of all polyhedral unions of equivalence classes. Given two equivalence relations $\mathcal{R}$ and $\mathcal{R}'$ over $S$ we say that $\mathcal{R}'$ is *coarser* than $\mathcal{R}$ or that $\mathcal{R}$ is *finer* than $\mathcal{R}'$ if $\mathcal{R} \subseteq \mathcal{R}'$.

**Definition 4.** *Let $\mathcal{A} = (L, L_0, X, E, AP, \mathcal{L}, (\mu_q, p_q)_{q \in L \times \mathbb{R}_+^X})$ be a STA. An equivalence relation $\mathcal{R}$ over $Q = L \times \mathbb{R}_+^X$ is a bisimulation for $\mathcal{A}$ if for all $q, q' \in Q$ with $q\mathcal{R}q'$: (i) $\mathcal{L}(q) = \mathcal{L}(q')$, and (ii) for every $I \in \mathcal{B}(\mathbb{R}_+)$, for every $C \in \mathrm{pcl}(\mathcal{R})$,*

$$\mathbb{P}_{\mathcal{A}}\big(\{q \models \xrightarrow{I,E} C\}\big) = \mathbb{P}_{\mathcal{A}}\big(\{q' \models \xrightarrow{I,E} C\}\big),$$

*where $\{q \models \xrightarrow{I,E} C\}$ stands for $\{\rho \in \mathrm{Runs}(\mathcal{A}, q) \mid \exists \tau \in I,\ \exists e \in E,\ \rho = q \xrightarrow{\tau,e} q_1 \to \cdots \wedge q_1 \in C\}$. States $q$ and $q'$ are bisimilar (written $q \sim q'$) if there is a bisimulation that contains $(q, q')$.*

Given $q \in Q$, $I \in \mathcal{B}(\mathbb{R}_+)$ and $C \in \mathrm{pcl}(\mathcal{R})$ the value $\mathbb{P}_{\mathcal{A}}\big(\{q \models \xrightarrow{I,E} C\}\big)$ can be expressed:

$$\mathbb{P}_{\mathcal{A}}\big(\{q \models \xrightarrow{I,E} C\}\big) = \int_{t \in I} P_{q+t}(C) f_q(t)\,\mathrm{d}t$$

where the value $P_{q+t}(C)$ corresponds to the probability to reach instantaneously $C$ from state $q+t$. Formally: $P_{q+t}(C) = \sum_{l' \in L} \sum_{e \in E_{l'}} p_{q+t}(e) \mathbb{1}_{\mathcal{C}_{l'}(e,\nu)}(t)$ for each

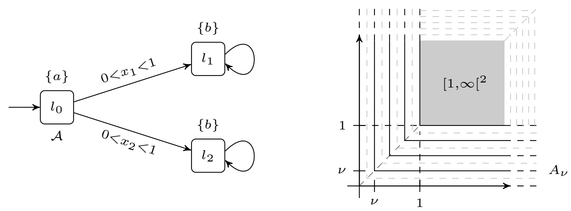


**Fig. 5.** A simple example for bisimulation.

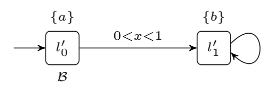


**Fig. 6.** $\mathcal{B}$ is bisimilar to $\mathcal{A}$.

$t \geq 0$ and each $C \in \mathrm{pcl}(\mathcal{R})$, where, given $l' \in L$, $E_{l'}$ is the set of edges with target $l'$, and given $e = (l, g, Y, l')$, $\mathcal{C}_{l'}(e, \nu) = \{t \in \mathbb{R}_+ \mid [Y \leftarrow 0](\nu + t) \in C_{l'}\}$. It can be shown that for every $t \geq 0$, $P_{q+t}$ is a probability measure over $Q$.

Also, given a STA $\mathcal{A}$, one can show that $\sim$ is the coarsest bisimulation for $\mathcal{A}$.

The above natural definition enjoys the following very nice characterization, which shows that our definition is conservative w.r.t. bisimulation over CTMCs [4,6].

**Proposition 2.** *Let $\mathcal{A}$ be a STA and let $\mathcal{R}$ be a bisimulation for $\mathcal{A}$. Then for all $q, q' \in Q$, $q\mathcal{R}q'$ if and only if (i)$\mathcal{L}(q) = \mathcal{L}(q')$, (ii) $\mu_q = \mu_{q'}$, and (iii) for every $C \in \mathrm{pcl}(\mathcal{R})$, $P_{q+t}(C) = P_{q'+t}(C)$ almost-surely for every $t \geq 0$.*

*Proof (Sketch).* Point (i) is obvious, and points (ii) and (iii) come from the fact that $q\mathcal{R}q'$ if for each $C \in \mathrm{pcl}(\mathcal{R})$ and for each $I \in \mathcal{B}(\mathbb{R}_+)$,

$$\int_{t \in I} P_{q+t}(C) f_q(t)\,\mathrm{d}t = \int_{t \in I} P_{q'+t}(C) f_{q'}(t)\,\mathrm{d}t.$$

With $C = L \times \mathcal{B}(\mathbb{R}_+^n)$, where $n$ is the number of clocks, we get that $P_{q+t}(C) = 1$ and thus $f_q = f_{q'}$ almost-surely, i.e. $\mu_q = \mu'_q$. It can then be easily shown that point (iii) holds. □

We now illustrate the notion of bisimulation on a simple example.

*Example 5.* Let us consider the simple STA $\mathcal{A}$ with two clocks on Figure 5. We assume exponential distributions with parameter $\lambda$ for every state at $l_1$ or $l_2$, and from a state of the form $q = (l_0, (\nu_1, \nu_2))$ with $\nu_1 < 1$ or $\nu_2 < 1$, $I(q) = [0, 1 - \min(\nu_1, \nu_2)[$ and so we can equip $q$ with a uniform distribution on the interval $I(q)$ for the delays.

The coarsest bisimulation $\sim$ can easily be computed and is shown on the right part of Figure 5: at location $l_0$, it is described by the following equivalence classes, for each $\nu \in [0, 1[$: $A_\nu = \{l_0\} \times \big(\{(\nu_1, \nu) \mid \nu_1 \geq \nu\} \cup \{(\nu, \nu_2) \mid \nu_2 \geq \nu\}\big)$.

We extend the previous notion of bisimulation to two STA in a standard way (see [7]), by considering the union of the two STA, and a bisimulation relation between the initial states. We postpone details to the Appendix, page 45. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are two STA, we write $\mathcal{A}_1 \sim \mathcal{A}_2$ when the two STA are bisimilar.

*Example 6.* Let us consider the one-clock STA $\mathcal{B}$ (Figure 6). Assuming that we have the same probability distributions as STA $\mathcal{A}$ of Figure 5, it can be easily established that $\mathcal{B} \sim \mathcal{A}$ by noticing that for each $\nu \in [0, 1[$, $(l'_0, \nu)$ is bisimilar to each state of $A_\nu$.

### 4.2 Congruence

One of the main objectives of defining behavioural equivalences is to aim at modular design and proof of correctness. This is only possible if bisimulation is a *congruence w.r.t. parallel composition*, that is, if $\mathcal{A}_1 \sim \mathcal{A}_2$, then for every $\mathcal{B}$, $\mathcal{A}_1 \parallel \mathcal{B} \sim \mathcal{A}_2 \parallel \mathcal{B}$. We first prove the following natural lemma which is a key point for proving the congruence of the bisimulation w.r.t. parallel composition. Though very intuitive, the result is surprisingly quite technical to prove.

**Lemma 2.** *Let $\mathcal{A}, \mathcal{B} \in$ CSTA$^*$ with sets of states resp. $Q_A$ and $Q_B$. If $\mathcal{R}$ is a bisimulation for $\mathcal{A}$ then the equivalence relation $\mathcal{R}'$ over $Q_A \times Q_B$ defined by $\mathcal{R}' = \{((q_1, q), (q_2, q)) \mid q_1 \mathcal{R} q_2$ and $q \in Q_B\}$, is a bisimulation for $\mathcal{A} \parallel \mathcal{B}$.*

We can now state the main result of this section:

**Theorem 3.** *Bisimulation is a congruence w.r.t. parallel composition. That is: if $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{B}$ are three STA in* CSTA$^*$, *if $\mathcal{A}_1 \sim \mathcal{A}_2$ then $\mathcal{A}_1 \parallel \mathcal{B} \sim \mathcal{A}_2 \parallel \mathcal{B}$.*

## 5 Conclusion

In this paper we have described a formal framework for compositional design of stochastic timed automata. We have established properties that should be satisfied by distributions over delays for well-defined parallel composition between components. We have proposed a natural notion of bisimulation and proven that it is a congruence w.r.t. parallel composition. We have also identified a subclass of STA which is closed under parallel composition.

We plan to extend our current work to so-called *interactive* STA (following [20,21]): the idea will be to add non-guarded interactive synchronizing events which take priority over delays when they are enabled. We hope that a parallel composition with synchronisation can be nicely defined in that setting, and that the model will enjoy nice properties as is the case in this paper.

There are many other plans for the future:

- Following the approach of [16,5], we would like to give a logical characterization of the bisimulation using (a subset of) CSL;
- We would like to be able, given a STA, to compute a small quotient automaton that would allow reduce the size of the system;
- All algorithms that have been developed so far for analyzing STA require a unique STA describing the system under analysis; we target the development of compositional verification (or approximation) methods, as it is done for instance for interactive Markov chains [12,22]. We would then like to see how this performs in practice.

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# A Details for Section 1

## A.1 STA as Modest-STA

This subsection is dedicated to readers who are familiar with the Modest-STA models [10].

In this subsection we explain how STA can be espressed as a Modest-STA. Let $\mathcal{A} = (L, L_0, X, E, AP, \mathcal{L}, (\mu_q, p_q)_{q \in L \times \mathbb{R}_+^X})$ be a STA.

We use a fresh float variable *delay* which will be used to sample delays according to the distributions, and a fresh clock $\mathsf{time}$, which will measure time elapsing in each location.

Pick $\ell \in L$, and assume that there are $n$ transitions leaving $\ell$, which are $\{e_i \mid 1 \leq i \leq n\}$, with $e_i = \ell \xrightarrow{g_i, Y_i} \ell_i$. In the constructed Modest-STA, we decouple the sampling of the delays, and the (probabilistic) choice of the transitions. The choice of the transitions will be made from location $\ell$, whereas the parameter *delay* will be sampled (in 0-delay) from the extra location $\widetilde{e_i}$. We have the following transitions:

- for every $\emptyset \neq I \subseteq \{1, \dots, n\}$,

  $$\ell \xrightarrow{\bigwedge_{i \in I} g_i \wedge \bigwedge_{i \notin I} \neg g_i \wedge \mathsf{time} = \mathit{delay}, \mathsf{time} \geq \mathit{delay}} \{(\mathrm{proba} = p_I(e_i), \widetilde{e_i}, \mathsf{time} := 0 \wedge Y_i := 0) \mid i \in I\}$$

  where $p_I(e_i)$ is the probability of taking edge $e_i$ in the original automaton, when exactly those transitions $e_j$ with $j \in I$ are enabled (that is, $\frac{w_i}{\sum_{j \in I} w_j}$ if probabilities are given by weights, assuming $w_j$ is the weight of edge $e_j$)
- for every $i \in I$,

  $$\widetilde{e_i} \xrightarrow{\mathsf{time}=0, \mathsf{time} \geq 0} (\mathrm{proba} = 1, \ell_i, \mathit{delay} := \mathrm{sample}(\mu_{\ell_i}(X)))$$

  where $\mu_{\ell_i}(X)$ is a description of the probability distributions over delays in location $\ell_i$.

A transition in a Modest-STA is of the form $\ell \xrightarrow{g,d} \{(p_i, a_i) \mid i \in I\}$ where $g$ is a guard that enables a transition and $d$ is a deadline (guard) ensuring that no more delay can elapse when $d$ is satisfied; $\sum_{i \in I} p_i = 1$ is a probability distribution, and $a_i$ is an assignment of clocks and variables, where standard variables (like *delay* here) can be sampled according to a distribution. Here $\mathrm{sample}(\mu_{\ell_i}(X))$ represents the sampling according to distributions over delays in $\ell_i$; this sampling depends on the clock values, and Modest will take the current valuation of clocks to evaluate that function. Of course, this requires distributions within one location are representable using functions definable in Modest (but no specific restrictive is put, except that it should be definable by a program terminating almost-surely).

When composing two such Modest-STA, the weak-memorylessness assumption will not be required; Indeed, the composition will increase the number of $\mathsf{time}$ and *delay* variables (one per original component), and we will build Modest-STA models that use more and more such variables; each *delay* will be sampled independently, and the races between components will be reflected by the order of the events “clock $\mathsf{time}_\alpha$ reaches its bound $delay_\alpha$”. But, even when the original STA satisfy the various hypotheses for safe composability, the constructed composed Modest-STA will no more reflect the fact that it is still a STA. Hence there will be no obvious way to import decidability technics developed for STA, e.g. in [8], to the corresponding fragment of Modest-STA.

## B Details for Section 2

**Lemma 3.** *Let $\mathcal{A}$ be a stochastic timed automaton. Then for each state $q$ of $\mathcal{A}$, $I(q)$ is either the empty set or a finite disjoint union of open intervals or intervals of the form $[0, a[$ with $a \in \mathbb{R}_+$.*

*Proof.* Let $\mathcal{A}$ be a stochastic timed automaton with $X = \{x_1, \ldots, x_n\}$ the set of clocks. We know that each guard $g$ appearing in $\mathcal{A}$ is of the form:

$$\bigwedge_{1 \le k \le m_1} (x_{i_k} < c_k) \wedge \bigwedge_{1 \le l \le m_2} (x_{j_l} > d_l),$$

where $m_1$, $m_2 \in \mathbb{N}$ and for each $k$ and $l$, $c_k$, $d_l \in \mathbb{N}$. Given such a guard $g$, we write $\mathrm{Sat}(g) = \{y \in \mathbb{R}_+^n \mid y \models g\}$ and thus it holds that

$$\mathrm{Sat}(g) = \bigcap_{1 \le k \le m_1} \mathrm{Sat}(x_{i_k} < c_k) \cap \bigcap_{1 \le l \le m_2} \mathrm{Sat}(x_{j_l} > d_l). \tag{B.1}$$

Now, let $q = (l, \nu)$ be a state of $\mathcal{A}$. We consider the function $f_\nu : \mathbb{R}_+ \to \mathbb{R}_+^n$ such that $f_\nu(t) = \nu + t$. It is easy to prove that for each $1 \le k \le m_1$ and for each $1 \le l \le m_2$,

- $f_\nu^{-1}(\mathrm{Sat}(x_{i_k} < c_k)) = [0, c_k - \nu_{i_k}[$ if $\nu_{i_k} < c_k$, it is the empty set otherwise,
- $f_\nu^{-1}(\mathrm{Sat}(x_{j_l} > d_l)) = ]d_l - \nu_{j_l}, \infty[$ if $\nu_{j_l} \le d_l$, and $f_\nu^{-1}(\mathrm{Sat}(x_{j_l} > d_l)) = [0 - \nu_{j_l}, \infty[$ otherwise.

From (B.1), we get that $f_\nu^{-1}(\mathrm{Sat}(g))$ is a finite intersection of such sets. It follows that $f_\nu^{-1}(\mathrm{Sat}(g))$ is either the empty set or a finite disjoint union of open intervals or intervals of the form $[0, a[$ with $a \in \mathbb{R}_+$. Finally, we write $G_l$ for the set of guards $g$ such that there is $e \in E$ with $e = (l, g, Y, l')$ for some $Y$ and $l'$. Then one can see that

$$I(q) = \bigcup_{g \in G_l} f_\nu^{-1}(\mathrm{Sat}(g)).$$

Since $G_l$ is a finite set, we conclude that $I(q)$ is either the empty set or a finite disjoint union of open intervals or intervals of the form $[0, a[$ with $a \in \mathbb{R}_+$. □

**Details for measuring constrained symbolic paths**

Given a STA $\mathcal{A}$, $q$ a state of $\mathcal{A}$, $e_1, \ldots, e_k$ edges of $\mathcal{A}$, and $\mathcal{C}$ a Borel set of $\mathbb{R}_+^k$, it holds that:

$$\mathbb{P}_{\mathcal{A}}(\pi_{\mathcal{C}}(q, e_1, \ldots, e_k)) = \int_{t_1 \in I(q,e_1)} p_{q+t_1}(e_1) \int_{t_2 \in I(q_{t_1}, e_2)} p_{q_{t_1}+t_2}(e_2) \ldots$$
$$\int_{t_k \in I(q_{t_1 \ldots t_{k-1}}, e_k)} p_{q_{t_1 \ldots t_{k-1}}+t_k}(e_k) \mathbb{1}_{\mathcal{C}}(t_1, \ldots, t_k) \, \mathrm{d}\mu_{q_{t_1 \ldots t_{k-1}}}(t_k) \ldots \mathrm{d}\mu_q(t_1)$$

where for every $i \geq 2$, the state $q_{t_1 \ldots t_i}$ is such that $q_{t_1 \ldots t_{i-1}} \xrightarrow{t_i} q_{t_1 \ldots t_{i-1}} + t_i \xrightarrow{e_i} q_{t_1 \ldots t_i}$ and the state $q_{t_1}$ is as defined before and $\mathbb{1}_{\mathcal{C}}(t_1, \ldots, t_k)$ is the characteristic function of $\mathcal{C}$.

**Lemma 4.** *Let $\mathcal{A} = (L, L_0, X, E, (\mu_q, p_q)_{q \in L \times \mathbb{R}_+^X})$ be a stochastic timed automaton and let $q$ be a state of $\mathcal{A}$. The set of Zeno runs starting from $q$ is an element of the $\sigma$-algebra generated by the cylinders. More precisely,*

$$\{\rho \in \mathrm{Runs}(\mathcal{A}, q) \mid \rho \textit{ is Zeno}\} = \bigcup_{M \in \mathbb{N}} \bigcap_{n \in \mathbb{N}_0} \bigcup_{(e_1, \ldots, e_n) \in E^n} \mathrm{Cyl}(\pi_{\mathcal{C}_{M,n}}(q, e_1, \ldots, e_n)),$$

*where* $\mathcal{C}_{M,n} = \{(\tau_1, \ldots, \tau_n) \in \mathbb{R}_+^n \mid \tau_1 + \ldots + \tau_n \leq M\}$.

*Proof.* If $\rho$ is a Zeno run starting from $q$, then there is some sequence of positive delays $(\tau_i)_{i \geq 1}$ and there is some sequence of edges $(e_i)_{i \geq 1}$ such that $\rho = q \xrightarrow{\tau_1, e_1} q_1 \xrightarrow{\tau_2, e_2} q_2 \cdots$ and $\sum_{i \geq 1} \tau_i < +\infty$. Thus, there is $M \in \mathbb{N}$ such that $\sum_{i \geq 1} \tau_i \leq M$. Hence for every $n \geq 1$, $\sum_{i=1}^n \tau_i \leq M$. This implies that for every $n$, $\rho \in \mathrm{Cyl}(\pi_{\mathcal{C}_{M,n}}(q, e_1, \ldots, e_n))$ and thus

$$\rho \in \bigcup_{M \in \mathbb{N}} \bigcap_{n \in \mathbb{N}_0} \bigcup_{(e_1, \ldots, e_n) \in E^n} \mathrm{Cyl}(\pi_{\mathcal{C}_{M,n}}(q, e_1, \ldots, e_n)).$$

Now, let us assume that there is $M \in \mathbb{N}$ such that for every $n \in \mathbb{N}_0$ there is $e_1, \ldots, e_n \in E$ such that $\rho \in \mathrm{Cyl}(\pi_{\mathcal{C}_{M,n}}(q, e_1, \ldots, e_n))$. Thus with

$$
\begin{array}{lll}
n = 1, & \rho = q \xrightarrow{\tau_1, e_1} q_1 \to \cdots & \text{s.t. } \tau_1 \leq M, \\
n = 2, & \rho = q \xrightarrow{\tau_1', e_1'} q_1' \xrightarrow{\tau_2, e_2} q_2 \to \cdots & \text{s.t. } \tau_1' + \tau_2 \leq M, \\
\vdots & & \\
n = k, & \rho = q \xrightarrow{\tau_1^{(k-1)}, e_1^{(k-1)}} q_1^{(k-1)} \to \cdots \xrightarrow{\tau_k, e_k} q_k \to \cdots & \text{s.t. } \sum_{i=1}^{k} \tau_i^{(k-i)} \leq M, \\
\vdots & &
\end{array}
$$

for some sequences of delays $\{(\tau_i^{(n)})_{n \in \mathbb{N}}\}_{i \geq 1}$ and some sequences of edges $\{(e_i^{(n)})_{n \in \mathbb{N}}\}_{i \geq 1}$, where $\tau_i^{(0)} = \tau_i$ and $e_i^{(0)} = e_i$ for each $i$. Now, one can easily see by induction that $\tau_i^{(n)} = \tau_i$ and $e_i^{(n)} = e_i$ for each $i$ and for each $n$. Thus, one can see that $\rho = q \xrightarrow{\tau_1, e_1} q_1 \xrightarrow{\tau_2, e_2} q_2 \xrightarrow{\tau_3, e_3} \cdots$ where for every $n \geq 1$, $\sum_{i=1}^{n} \tau_i \leq M$. This implies that $\rho$ is Zeno. □

The almost-surely (non-)Zenoness of a timed automaton can thus be expressed as follows.

**Proposition 3.** *Let $\mathcal{A} = \big(L, L_0, X, E, AP, \mathcal{L}, (\mu_q, p_q)_{q \in L \times \mathbb{R}_+^X}\big)$ be a stochastic timed automaton, we have that*

(i) *$\mathcal{A}$ is almost-surely Zeno iff for each initial state $q_0$ of $\mathcal{A}$*

$$\lim_{M \to +\infty} \lim_{n \to +\infty} \sum_{(e_1, \ldots, e_n) \in E^n} \mathbb{P}_{\mathcal{A}}(\mathrm{Cyl}(\pi_{\mathcal{C}_{M,n}}(q_0, e_1, \ldots, e_n))) = 1,$$

(ii) *$\mathcal{A}$ is almost-surely non-Zeno iff for each initial state $q_0$ of $\mathcal{A}$, for every $M \in \mathbb{N}$,*

$$\lim_{n \to +\infty} \sum_{(e_1, \ldots, e_n) \in E^n} \mathbb{P}_{\mathcal{A}}(\mathrm{Cyl}(\pi_{\mathcal{C}_{M,n}}(q_0, e_1, \ldots, e_n))) = 0.$$

*Proof.* For each $M \in \mathbb{N}$ and for each $n \in \mathbb{N}_0$, we write:

$$Y_{n,M} = \bigcup_{(e_1, \ldots, e_n) \in E^n} \mathrm{Cyl}(\pi_{\mathcal{C}_{M,n}}(q, e_1, \ldots, e_n)) \quad \text{and} \quad X_M = \bigcap_{n \in \mathbb{N}_0} Y_{n,M}.$$

Thus, from Lemma 4 one can see that $\mathcal{A}$ is almost-surely Zeno if and only if

$$\mathbb{P}_{\mathcal{A}}\Big(\bigcup_{M \in \mathbb{N}} X_M\Big) = 1. \tag{B.2}$$

Now from classical results of measure theory, since

$$X_M \underset{M \to +\infty}{\nearrow} \bigcup_{M \in \mathbb{N}} X_M \quad \text{and} \quad Y_{n,M} \underset{n \to +\infty}{\searrow} \bigcap_{n \in \mathbb{N}_0} Y_{n,M}$$

(one can easily check that $X_M \subseteq X_{M+1}$ and $Y_{n+1,M} \subseteq Y_{n,M}$ for every $M$ and $n$), one can see that

$$\begin{aligned}
\mathbb{P}_{\mathcal{A}}\Big(\bigcup_{M \in \mathbb{N}} X_M\Big) &= \lim_{M \to +\infty} \mathbb{P}_{\mathcal{A}}(X_M) \\
&= \lim_{M \to +\infty} \mathbb{P}_{\mathcal{A}}\Big(\bigcap_{n \in \mathbb{N}_0} Y_{n,M}\Big) \\
&= \lim_{M \to +\infty} \lim_{n \to +\infty} \mathbb{P}_{\mathcal{A}}(Y_{n,M}) \\
&= \lim_{M \to +\infty} \lim_{n \to +\infty} \mathbb{P}_{\mathcal{A}}\Big(\bigcup_{(e_1, \ldots, e_n) \in E^n} \mathrm{Cyl}(\pi_{\mathcal{C}_{M,n}}(q, e_1, \ldots, e_n))\Big) \\
&= \lim_{M \to +\infty} \lim_{n \to +\infty} \sum_{(e_1, \ldots, e_n) \in E^n} \mathbb{P}_{\mathcal{A}}(\mathrm{Cyl}(\pi_{\mathcal{C}_{M,n}}(q_0, e_1, \ldots, e_n))),
\end{aligned} \tag{B.3}$$

since two cylinders of same length that differ in the edges are disjoint. From (B.2) and (B.3), we deduce (i).

Similarly, we have that $\mathcal{A}$ is almost-surely non-Zeno if and only if

$$\mathbb{P}_{\mathcal{A}}\Big(\bigcup_{M\in\mathbb{N}} X_M\Big) = 0.$$

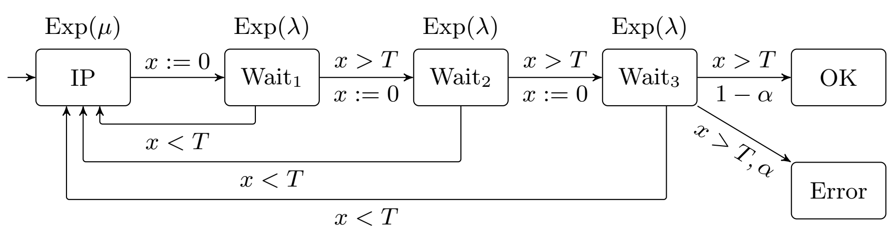
This equality holds if and only if for every $M \in \mathbb{N}$, $\mathbb{P}_{\mathcal{A}}(X_M) = 0$. From (B.3) we thus deduce (ii). $\square$

*Remark 6.* Let us observe that, given $M \in \mathbb{N}$, $n \in \mathbb{N}_0$ and $e_1, \ldots, e_n \in E$ one can express $\mathbb{P}_{\mathcal{A}}(\mathrm{Cyl}(\pi_{\mathcal{C}_{M,n}}(q_0, e_1, \ldots, e_n)))$ as the result of the following $n$-dimensional integral:

$$\begin{aligned}
&\mathbb{P}_{\mathcal{A}}(\mathrm{Cyl}(\pi_{\mathcal{C}_{M,n}}(q_0, e_1, \ldots, e_n))) \\
&\quad= \int_{t_1=0}^{M} p_{q_0+t_1}(e_1)\mathbb{1}_{I(q_0,e_1)}(t_1) \int_{t_2=0}^{M-t_1} p_{q_{t_1}+t_2}(e_2)\mathbb{1}_{I(q_{t_1},e_2)}(t_2)\ldots \\
&\quad\int_{t_n=0}^{M-t_1-\ldots t_{n-1}} p_{q_{t_1\ldots t_{n-1}}+t_n}(e_n)\mathbb{1}_{I(q_{t_1\ldots t_{n-1}},e_n)}(t_n)\,\mathrm{d}\mu_{q_{t_1\ldots t_{n-1}}}(t_n)\ldots\mathrm{d}\mu_{q_0}(t_1).
\end{aligned}$$

**Details for Example 1**

We recall the IPv4 Zeroconf protocol example depicted in Figure 1 (repeated below). The task of this protocol is to configure IP addresses in local network of appliances in the following way. When a new appliance is connected to the network, a unique IP address has to be configured with this appliance. The protocol selects randomly an IP address and then sends $N$ messages to the network in order to verify if the IP address is already used. If one of the messages receives an answer in a bounded time, then the IP address is already used and a new one is selected. If none of the messages get an answer, then either the IP address is not used and the appliance is well plugged, or the IP address is used and there was an error when sending the messages.




The IPv4 Zeroconf STA for $N = 3$.

In Figure 1, we model the protocol for $N = 3$ as a stochastic timed automaton with a single clock $x$. We can describe it as follows. In the initial location IP, an

IP address has been randomly selected. Then a first message is broadcast and we go to location $\text{Wait}_1$ after resetting clock $x$ to 0. From there, either an answer is received before $T$ time units and we go back to location IP where a new IP address is randomly chosen, or no answer is given before $T$ time units and then a new message is sent, we reset $x$ to 0 and we go to location $\text{Wait}_2$ where we observe a similar behaviour. In location $\text{Wait}_3$ the third and last message has been broadcast. Then again, either an answer is received before $T$ time units and we return to location IP, or no answer is sent. If there was an error in the broadcast of the messages and the IP address is already used, we move to location Error, otherwise we move to location OK.

We finally have to equip each state of probability distributions over the edges and the delays. From state $(\text{IP}, \nu)$ with $\nu \in \mathbb{R}_+$, $I((\text{IP}, \nu)) = \mathbb{R}_+$ so that we could equip the state with an exponential distribution of parameter $\mu$, written $\text{Exp}(\mu)$, over the delays. And since there is only one edge enabled, the probability over the edges is trivial. For states of the form $(\text{Wait}_i, \nu)$, the set of delays is $\mathbb{R}_+ \setminus \{T - \nu\}$ so that we can define the probability over the delays in $(\text{Wait}_i, \nu)$ as $\text{Exp}(\lambda)$. Now, if $\nu < T$, there is only one enabled edge and so the probability over the edges is trivial. If $\nu = T$, then no edge is enabled. Finally, if $\nu > T$ then there is a unique enabled edge in $(\text{Wait}_1, \nu)$ and $(\text{Wait}_2, \nu)$ so that the probability is trivial. In state $(\text{Wait}_3, \nu)$, we can either move to OK or to Error, and we assume that the probability to go to Error is given by $0 < \alpha < 1$.

**The G/G/1/$k$-queue example**

We consider a queuing system in which jobs arrive and wait until they are executed, and we assume the capacity of the queue is $k$. We assume the interarrival time is given by distribution $\Lambda_{it}$, and that the process time of each job is given by distribution $\Lambda_{pt}$. We will use $f$'s for density functions and $F$'s for cumulative functions, with respective indices. This is what is known as a G/G/1/$k$-queue (the two first $G$'s stand for "Generalized" distributions for arrival times and process times).

We will propose an STA model for this queuing system. To do so we need to use some of the transformations that appear later in the paper. However we believe this can be understood without going to the details of the paper first.

- For every $\nu_x \geq 0$, we write $\Lambda_{it,\nu_x}$ for the distribution obtained from $\Lambda_{it}$ by conditioning over the fact that the job has not arrived within the first $\nu_x$ time units. It is obtained using rule (B) on page 7 and is expressed as follows:

$$f_{it,\nu_x}(t) = \frac{f_{it}(\nu_x + t)}{1 - F_{it}(\nu_x)}$$

  Note that this is only defined for $\nu_x$ smaller than the upper bound of the support of $\Lambda_{it}$.
- Similarly, for every $\nu_y \geq 0$, we write $\Lambda_{pt,\nu_y}$ for the distribution obtained from $\Lambda_{pt}$ by conditioning over the fact that the job has not arrived within the first $\nu_x$ time units. The expression is identical to the first item.

- For every $\nu_x, \nu_y \geq 0$, we write $\Lambda_{\min,(\nu_x,\nu_y)}$ or simply $\Lambda_{\min,\nu}$ for the distribution $\min(\Lambda_{it,\nu_x}, \Lambda_{pt,\nu_y})$, representing a race between the two distributions representing arrival of job, and processing of job. It can be computed using the technical developments page 8 and can be expressed as follows:

$$f_{\min,\nu}(t) = f_{it,\nu_x}(t) \cdot (1 - F_{pt,\nu_y}(t)) + f_{pt,\nu_y}(t) \cdot (1 - F_{it,\nu_x}(t))$$

- For every $\nu_x, \nu_y \geq 0$, we define probabilities $p^x_{(\nu_x,\nu_y)}(t)$ and $p^y_{(\nu_x,\nu_y)}(t)$ (or simply $p^x_\nu(t)$ and $p^y_\nu(t)$) for the probabilities of having a job arrival (resp. processing) at that time, under the assumption that the delay is $t$ since one arrived in the location with valuation $\nu = (\nu_x, \nu_y)$. Following the rules described on page 8, it can be expressed as follows:

$$p^x_\nu(t) = \frac{f_{it,\nu_x}(t) \cdot (1 - F_{pt,\nu_y}(t))}{f_{\min,\nu}(t)} \text{ and } p^y_\nu(t) = \frac{f_{pt,\nu_y}(t) \cdot (1 - F_{it,\nu_x}(t))}{f_{\min,\nu}(t)}$$

  Note that it is the case that $p^\star_\nu(t) = p^\star_{\nu'}(t')$ whenever $\nu + t = \nu' + t'$.

The STA for the G/G/1/$k$-queue is now depicted in Figure 7.

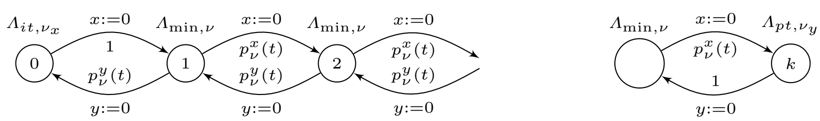



**Fig. 7.** A G/G/1/$k$-queue

# C Details for Section 3

## C.1 Details for Subsection 3.1

In the sequel, let $q = (q_1, q_2)$ be a state of the product $\mathcal{A}_1 \parallel \mathcal{A}_2$, let $\mathbb{X}_{q_1}$ (resp. $\mathbb{X}_{q_2}$) be a random variable of density function $f_{q_1}$ (resp. $f_{q_2}$), with $\mathbb{X}_{q_1}$ and $\mathbb{X}_{q_2}$ independent and $f_{q_1}$ and $f_{q_2}$ continuous everywhere except in finite number of points.

**Lemma 5.** *It holds that* $\min(\mathbb{X}_{q_1}, \mathbb{X}_{q_2})$ *is a random variable of density function* $f_q$ *defined by* $f_q(t) = f_{q_1}(t)(1 - F_{q_2}(t)) + f_{q_2}(t)(1 - F_{q_1}(t))$ *for almost every* $t \geq 0$.

*Proof.* Let $f_q$ be the density function of the random variable $\min(\mathbb{X}_{q_1}, \mathbb{X}_{q_2})$. Then the cumulative function of $\min(\mathbb{X}_{q_1}, \mathbb{X}_{q_2})$ is defined by $F_q(t) = \mathbb{P}\big(\min(\mathbb{X}_{q_1}, \mathbb{X}_{q_2}) \leq t\big)$ for every $t \geq 0$ and thus $1 - F_q(t) = \mathbb{P}\big(\min(\mathbb{X}_{q_1}, \mathbb{X}_{q_2}) \geq t\big)$ for every $t \geq 0$. Now, since

$$\begin{aligned}\mathbb{P}\big(\min(\mathbb{X}_{q_1}, \mathbb{X}_{q_2}) \geq t\big) &= \mathbb{P}\big(\{\mathbb{X}_{q_1} \geq t\} \cap \{\mathbb{X}_{q_2} \geq t\}\big) \\ &= \mathbb{P}(\mathbb{X}_{q_1} \geq t)\mathbb{P}(\mathbb{X}_{q_2} \geq t) \quad \text{by independence of } \mathbb{X}_{q_1} \text{ and } \mathbb{X}_{q_2} \\ &= (1 - F_{q_1}(t))(1 - F_{q_2}(t)),\end{aligned}$$

we deduce that $(1 - F_q(t)) = (1 - F_{q_1}(t))(1 - F_{q_2}(t))$ for every $t \geq 0$ and thus, when we compute the derivative of this last equality we get that $f_q(t) = f_{q_1}(t)(1 - F_{q_2}(t)) + f_{q_2}(t)(1 - F_{q_1}(t))$ for every $t \geq 0$ such that $f_{q_1}$ and $f_{q_2}$ are continuous in $t$, i.e. for every $t \geq 0$ except a finite number of points. ☐

Let $w_q^1(t) = \lim_{\varepsilon \to 0} \mathbb{P}\big(\mathbb{X}_{q_1} = \min(\mathbb{X}_{q_1}, \mathbb{X}_{q_2}) \mid \min(\mathbb{X}_{q_1}, \mathbb{X}_{q_2}) \in [t, t+\varepsilon]\big)$ for every $t \in I(q)$.

**Lemma 6.** *It holds that for every $t \in I(q)$ except a finite number of points, $w_q^1(t) = \frac{f_{q_1}(t)(1-F_{q_2}(t))}{f_q(t)}$.*

*Proof.* For any $t$ in $I(q)$ we have that

$$\begin{aligned} w_q^1(t) &:= \lim_{\varepsilon \to 0} \mathbb{P}\big(\mathbb{X}_{q_1} = \min(\mathbb{X}_{q_1}, \mathbb{X}_{q_2}) \mid \min(\mathbb{X}_{q_1}, \mathbb{X}_{q_2}) \in [t, t+\varepsilon]\big) \\ &= \lim_{\varepsilon \to 0} \frac{\frac{1}{\varepsilon}\mathbb{P}\big(\mathbb{X}_{q_1} = \min(\mathbb{X}_{q_1}, \mathbb{X}_{q_2}) \wedge \min(\mathbb{X}_{q_1}, \mathbb{X}_{q_2}) \in [t, t+\varepsilon]\big)}{\frac{1}{\varepsilon}\mathbb{P}\big(\min(\mathbb{X}_{q_1}, \mathbb{X}_{q_2}) \in [t, t+\varepsilon]\big)}. \end{aligned}$$

Since $t \in I(q)$, from Lemma 3 we get that there is $\delta > 0$ such that for every $\varepsilon < \delta$, $[t, t+\varepsilon] \subseteq I(q)$ and thus $\mathbb{P}\big(\min(\mathbb{X}_{q_1}, \mathbb{X}_{q_2}) \in [t, t+\varepsilon]\big) \neq 0$. We can thus compute $w_q^1(t)$ as follows. We have

$$\begin{aligned} &\mathbb{P}\big(\mathbb{X}_{q_1} = \min(\mathbb{X}_{q_1}, \mathbb{X}_{q_2}) \wedge \min(\mathbb{X}_{q_1}, \mathbb{X}_{q_2}) \in [t, t+\varepsilon]\big) \\ &\quad = \mathbb{P}\big(\mathbb{X}_{q_1} = \min(\mathbb{X}_{q_1}, \mathbb{X}_{q_2}) \wedge \mathbb{X}_{q_1} \in [t, t+\varepsilon]\big) \\ &\quad = \int_{t_1=t}^{t+\varepsilon} \int_{t_2=t_1}^{+\infty} f_{q_2}(t_2) f_{q_1}(t_1) \, \mathrm{d}t_2 \, \mathrm{d}t_1 \\ &\qquad \text{by classical results of probability theory, since } \mathbb{X}_{q_1} \text{ and } \mathbb{X}_{q_2} \text{ are independent} \\ &\quad = \int_{t_1=t}^{t+\varepsilon} f_{q_1}(t_1)(1 - F_{q_2}(t_1)) \, \mathrm{d}t_1 \quad \text{by definition of the cumulative function} \end{aligned} \tag{C.1}$$

for every $t$ in $I(q)$. Let us denote $g(t_1) = f_{q_1}(t_1)(1 - F_{q_2}(t_1))$ for every $t_1 \geq 0$. Since $f_{q_1}$ and $f_{q_2}$ are continuous everywhere except in a finite nuber of points, it holds that $g$ is also continuous on $\mathbb{R}_+$ except in a finite number of points. Let us assume that $g$ is continuous in $t$. Then, it can be supposed that $g$ is continuous on $[t, t+\varepsilon]$ and since it is a closed interval, we have that $g$ reaches its bounds on $[t, t+\varepsilon]$. We have

$$\int_{t_1=t}^{t+\varepsilon} \min_{x \in [t, t+\varepsilon]} g(x) \, \mathrm{d}t_1 \leq \int_{t_1=t}^{t+\varepsilon} g(t_1) \, \mathrm{d}t_1 \leq \int_{t_1=t}^{t+\varepsilon} \max_{x \in [t, t+\varepsilon]} g(x) \, \mathrm{d}t_1,$$

and thus,

$$\min_{x \in [t, t+\varepsilon]} g(x) \leq \frac{1}{\varepsilon} \int_{t_1=t}^{t+\varepsilon} g(t_1) \, \mathrm{d}t_1 \leq \max_{x \in [t, t+\varepsilon]} g(x). \tag{C.2}$$

Now, since $g$ is continuous on $[t, t+\varepsilon]$, we have

$$\min_{x \in [t, t+\varepsilon]} g(x) \underset{\varepsilon \to 0}{\longrightarrow} g(t) \quad \text{and} \quad \max_{x \in [t, t+\varepsilon]} g(x) \underset{\varepsilon \to 0}{\longrightarrow} g(t).$$

From (C.1) and (C.2), we thus deduce that

$$\frac{1}{\varepsilon}\mathbb{P}\big(\mathbb{X}_{q_1} = \min(\mathbb{X}_{q_1}, \mathbb{X}_{q_2}) \wedge \min(\mathbb{X}_{q_1}, \mathbb{X}_{q_2}) \in [t, t+\varepsilon]\big) \underset{\varepsilon \to 0}{\longrightarrow} f_{q_1}(t)(1-F_{q_2}(t)) \quad \text{(C.3)}$$

almost-surely for every $t \in I(q)$ (it holds for every $t$ such that $f_{q_1}$ is continuous in $t$). Similarly, we have that

$$\begin{aligned}
&\frac{1}{\varepsilon}\mathbb{P}\big(\min(\mathbb{X}_{q_1}, \mathbb{X}_{q_2}) \in [t, t+\varepsilon]\big) \\
&\quad = \frac{1}{\varepsilon}\big(\mathbb{P}(\mathbb{X}_{q_1} \in [t, t+\varepsilon] \wedge \mathbb{X}_{q_1} \leq \mathbb{X}_{q_2}) + \mathbb{P}(\mathbb{X}_{q_2} \in [t, t+\varepsilon] \wedge \mathbb{X}_{q_2} \leq \mathbb{X}_{q_1})\big) \\
&\quad \underset{\varepsilon \to 0}{\longrightarrow} f_{q_1}(t)(1 - F_{q_2}(t)) + f_{q_2}(t)(1 - F_{q_1}(t)) = f_q(t), \qquad \text{(C.4)}
\end{aligned}$$

for every $t \in I(q)$ such that $f_{q_1}$ and $f_{q_2}$ are continuous in $t$, i.e. for every $t \in I(q)$ except in a finite number of points. Finally, it should be noted that since $f_q$ is equivalent to the restriction of the Lebesgue measure on $I(q)$, one can assume w.l.o.g. that $f_q(t) \neq 0$ for each $t \in I(q)$. Hence, from (C.3) and (C.4) we deduce that

$$w_q^1(t) = \frac{f_{q_1}(t)(1 - F_{q_2}(t))}{f_q(t)}$$

for every $t$ in $I(q)$ except in a finite number of points. □

We recall the definition of the class CSTA: we say that $\mathcal{A} \in$ CSTA if for every state $q$ of $\mathcal{A}$,

(A) the density function associated with $\mu_q$, denoted by $f_q$, is continuous everywhere on $\mathbb{R}_+$ except in a finite number of points, and
(B) the probability distribution $\mu_q$ is *weakly-memoryless*, i.e. for every $t, t' \geq 0$, $\mathbb{P}_{\mathcal{A}}(\mathbb{X}_q \geq t + t' \mid \mathbb{X}_q \geq t) = \mathbb{P}_{\mathcal{A}}(\mathbb{X}_{q+t} \geq t')$, where $\mathbb{X}_q$ (resp. $\mathbb{X}_{q+t}$) is a random variable of the density function $f_q$ (resp. $f_{q+t}$)

**Lemma 7.** *The probability distribution $\mu_q$ is weakly-memoryless iff for every $t$, $t' \geq 0$,*

$$f_q(t + t') = (1 - F_q(t))f_{q+t}(t') \qquad (1)$$

*except in a finite number of points.*

*Proof.* First, it should be noted that $\mathbb{P}_{\mathcal{A}}(\mathbb{X}_q \geq t) = 0$ iff $I(q + t) = \emptyset$. This comes from Lemma 3. It is thus not possible to leave state $q$ after $t + t'$ time units for each $t' \geq 0$, so that such $t$ do not have to be considered. Now let $t \geq 0$ be such that $\mathbb{P}_{\mathcal{A}}(\mathbb{X}_q \geq t) > 0$. Then,

$$\begin{aligned}
&\mathbb{P}_{\mathcal{A}}(\mathbb{X}_q \geq t + t' \mid \mathbb{X}_q \geq t) = \mathbb{P}_{\mathcal{A}}(\mathbb{X}_{q+t} \geq t') \\
&\quad \Leftrightarrow \frac{\mathbb{P}_{\mathcal{A}}(\mathbb{X}_q \geq t + t', \mathbb{X}_q \geq t)}{\mathbb{P}_{\mathcal{A}}(\mathbb{X}_q \geq t)} = \mathbb{P}_{\mathcal{A}}(\mathbb{X}_{q+t} \geq t') \\
&\quad \Leftrightarrow (1 - F_q(t + t')) = (1 - F_q(t))(1 - F_{q+t}(t')) \qquad \text{(C.5)} \\
&\quad \Leftrightarrow \partial_{t'}(1 - F_q(t + t')) = \partial_{t'}\big((1 - F_q(t))(1 - F_{q+t}(t'))\big) \\
&\quad \Leftrightarrow f_q(t + t') = (1 - F_q(t))f_{q+t}(t')
\end{aligned}$$

for every $t' \geq 0$ in which $f_{q+t}(t')$ and $f_q(t+t')$ are continuous, i.e. everywhere except in a finite number of points. □

*Remark 7.* Note that since $f_q$ and $f_{q+t}$ are density functions, we can assume w.l.o.g. that (1) holds for every $t$ and $t' \geq 0$.

### C.2 Details for Subsection 3.2

Assuming the notations of Definition 3, we prove:

**Lemma 1.** *The distributions $\mu_q$ and $p_q$ are well-defined, and the STA $\mathcal{A}_1 \parallel \mathcal{A}_2$ belongs to the class* CSTA*.*

*Proof.* We can show that parallel composition is well-defined and internal in CSTA. We should first make clear what we mean by "well-defined". Let $\mathcal{A}_1$ and $\mathcal{A}_2 \in$ CSTA, in order to construct $\mathcal{A}_1 \parallel \mathcal{A}_2$, we have defined probability distributions over both delays and edges from each state $q$ of the product. It can be easily established that $\mu_q$ and $p_q$ are probability distributions over the delays and edges. For the delays, this is directly ensured from the construction of $f_q$. For the edges, we have to show that for each $t \in I(q)$, $p_{q+t}$ defines a probability distribution over the set of enabled edges and that for each $e \in E$, $p_{q+\bullet}(e)$ is measurable. This last statement is trivial since $p_{q_1+\bullet}(e_1)$ (resp. $p_{q_2+\bullet}(e_2)$) is measurable for each $e_1 \in E_1$ (resp. $e_2 \in E_2$) and since $w_q^1$ and $w_q^2$ are continuous on $\mathbb{R}_+$ except in a finite number of points. Now in order to get that $p_{q+t}$ defines a probability distribution over the set of enabled edges for each $t \in I(q)$, we have that

$$\begin{aligned}
&\{((l_1, l_2), g, Y, (l'_1, l'_2)) \in E \mid (\nu_1 + t, \nu_2 + t) \models g\} \\
&\quad = \{(l_1, g, Y, l'_1) \in E_1 \mid \nu_1 + t \models g\} \cup \{(l_2, g, Y, l'_2) \in E_2 \mid \nu_2 + t \models g\} \quad \text{(C.6)}
\end{aligned}$$

where $q = ((l_1, l_2), (\nu_1, \nu_2))$, and for any $t \in I(q)$,

$$\begin{aligned}
w_q^1(t) + w_q^2(t) &= \frac{f_{q_1}(t)(1 - F_{q_2}(t))}{f_q(t)} + \frac{f_{q_2}(t)(1 - F_{q_1}(t))}{f_q(t)} \\
&= \frac{f_{q_1}(t)(1 - F_{q_2}(t)) + f_{q_2}(t)(1 - F_{q_1}(t))}{f_q(t)} \\
&= 1 \quad \text{by definition of } f_q(t). \qquad \text{(C.7)}
\end{aligned}$$

Now if $t \in I(q_1)$ and $t \notin I(q_2)$, then $w_q^2(t) = 0$ and thus $w_q^1(t) = 1$ from (C.7). Thus,

$$\sum_{e \in E} p_{q+t}(e) = \sum_{e \in E_1} p_{q_1+t}(e) = 1,$$

since $p_{q_1+t}$ is a probability distribution over the set of (enabled) edges in $E_1$, and thus $p_{q+t}$ is a probability distribution over $E$. Similarly we have, if $t \notin I(q_1)$ and

$t \in I(q_2)$, that $\sum_{e \in E} p_{q+t}(e) = 1$ and thus $p_{q+t}$ is a probability distribution over $E$. Now, if $t \in I(q_1) \cap I(q_2)$, then from (C.6) and (C.7), we have that

$$\begin{aligned}\sum_{e \in E} p_{q+t}(e) &= \sum_{e \in E_1} p_{q+t}(e) + \sum_{e \in E_2} p_{q+t}(e) \\ &= w_q^1(t) \sum_{e \in E_1} p_{q_1+t}(e) + w_q^2(t) \sum_{e \in E_2} p_{q_2+t}(e) \\ &= w_q^1(t) + w_q^2(t) = 1.\end{aligned}$$

Hence, for any $t \in I(q)$, $p_{q+t}$ is a probability distribution over the set of enabled edges in $q + t$.

Now, for the delays we have defined $\mu_q$ for each state $q$ separately so that no problems could be encountered. However for the edges, from each state $q$ we have defined the probability $p_{q+t}$ for each $t \geq 0$. Hence, $\mathcal{A}_1 \parallel \mathcal{A}_2$ is well-defined only if for each state $q$ and for each $t, t' \geq 0$, $p_{q+(t+t')} = p_{(q+t)+t'}$. This equality holds if $w_q^i(t + t') = w_{q+t}^i(t')$ for any $t, t' \geq 0$ and for any $i \in \{1, 2\}$, which is ensured thanks to hypothesis (B). Indeed, we have

$$\begin{aligned}& w_q^1(t + t') = w_{q+t}^1(t') \\ &\quad \Leftrightarrow \frac{f_{q_1}(t + t')(1 - F_{q_2}(t + t'))}{f_q(t + t')} = \frac{f_{q_1+t}(t')(1 - F_{q+t}(t'))}{f_{q+t}(t')} \\ &\quad \Leftrightarrow f_{q_1}(t + t')(1 - F_{q_2}(t + t'))f_{q_2+t}(t')(1 - F_{q_1+t}(t')) \\ &\qquad = f_{q_2}(t + t')(1 - F_{q_1}(t + t'))f_{q_1+t}(t')(1 - F_{q+t}(t')).\end{aligned}$$

We show that for each $i = 1, 2$, and for each $t$ and $t' \geq 0$, $f_{q_i}(t+t')(1-F_{q_i+t}(t')) = f_{q_i+t}(t')(1-F_{q_i}(t+t'))$. Let $t, t' \geq 0$, from the equivalences in (C.5) we have that

$$\begin{aligned}& f_{q_i}(t + t') = (1 - F_{q_i}(t))f_{q_i+t}(t') \\ &\quad \Leftrightarrow (1 - F_{q_i}(t + t')) = (1 - F_{q_i}(t))(1 - F_{q_i+t}(t')).\end{aligned} \tag{C.5}$$

First, let us notice that if $(1 - F_{q_i+t}(t')) = 0$, then $(1 - F_{q_i}(t + t')) = 0$ from the last equivalence and since $\mathcal{A}_i \in$ CSTA. Thus $f_{q_i}(t + t')(1 - F_{q_i+t}(t')) = f_{q_i+t}(t')(1-F_{q_i}(t+t'))$. Now if $(1-F_{q_i+t}(t')) \neq 0$, under hypothesis (B) and (C.5), we have that

$$\begin{aligned}& f_{q_i}(t + t')(1 - F_{q_i+t}(t')) = f_{q_i+t}(t')(1 - F_{q_i}(t + t')) \\ &\quad \Leftrightarrow f_{q_i}(t + t') = f_{q_i+t}(t')\frac{(1 - F_{q_i}(t + t'))}{(1 - F_{q_i+t}(t'))} \\ &\quad \Leftrightarrow f_{q_i}(t + t') = (1 - F_{q_i}(t))f_{q_i+t}(t')\end{aligned}$$

which is true. We conclude that $w_q^1(t + t') = w_{q+t}^1(t')$ for every $t$ and $t' \geq 0$, which ensures us that $p_q$ is well-defined.

It remains to show that $\mathcal{A}_1 \parallel \mathcal{A}_2 \in$ CSTA: we need to show that for each state $q$ of the product, $f_q$ satisfies hypotheses (A) and (B). Let $q$ be a state of

$\mathcal{A}_1 \parallel \mathcal{A}_2$. Point (A) is easily established since $f_{q_1}$ and $f_{q_2}$ satisfy this point and since $F_{q_1}$ and $F_{q_2}$ are continuous. Now let $t, t' \geq 0$, from (C.5), we have that

$$\begin{aligned} & f_q(t+t') = (1 - F_q(t)) f_{q+t}(t') \\ & \quad \Leftrightarrow (1 - F_q(t+t')) = (1 - F_q(t))(1 - F_{q+t}(t')). \end{aligned}$$

Now by recalling that $(1 - F_q(t+t')) = \mathbb{P}(\{\mathbb{X}_{q_1} \geq t+t'\} \cap \{\mathbb{X}_{q_2} \geq t+t'\})$ with $\mathbb{X}_{q_1}$ and $\mathbb{X}_{q_2}$ independent, we can compute $(1 - F_q(t+t'))$ as follows:

$$\begin{aligned} (1 - F_q(t+t')) &= (1 - F_{q_1}(t+t'))(1 - F_{q_2}(t+t')) \\ &= (1 - F_{q_1}(t))(1 - F_{q_1+t}(t'))(1 - F_{q_2}(t))(1 - F_{q+t}(t')) \\ &\quad \text{by hypotheses over } \mathcal{A}_1 \text{ and } \mathcal{A}_2 \\ &= \big((1 - F_{q_1}(t))(1 - F_{q_2}(t))\big)\big((1 - F_{q_1+t}(t'))(1 - F_{q+t}(t'))\big) \\ &= (1 - F_q(t))(1 - F_{q+t}(t')) \end{aligned}$$

which is what we want. □

**Details for Example 2**

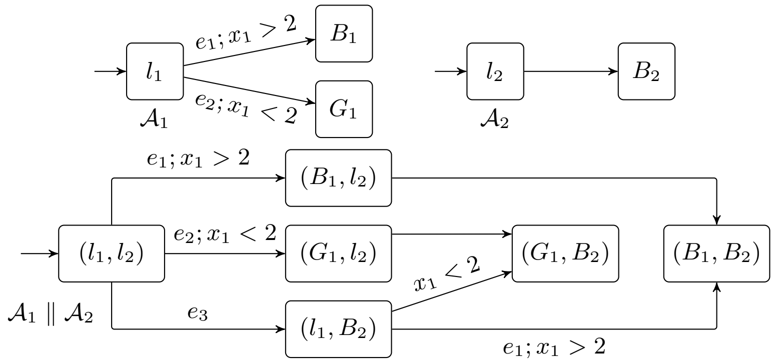
We consider the single-clock STA $\mathcal{A}_1$ depicted in Figure 2 (repeated below). We assume $\mu^{(1)}_{q_1}$ is an exponential distribution of parameter $\lambda_1$ (resp. $\lambda_2$) if $q_1 = (l_1, \nu_1)$ with $\nu_1 < 1$ (resp. $\nu_2 \geq 1$), and with $\lambda_1 \neq \lambda_2$. Then for each $\nu_1 \in [0, 1[$, $\mu^{(1)}_{q_1}$ does not satisfy hypothesis (B). We assume that from $l_1$, $\mathcal{A}_1$ can move to $G_1$ if $x_1 < 2$ or to $B_1$ if $x_1 > 2$, and from there, it stays in the same location with probability 1. We then compose $\mathcal{A}_1$ (using Definition 3) with the single-clock STA $\mathcal{A}_2$, where $\mathcal{A}_2$ has $l_2$ as its initial location from which it can move at any time to $B_2$ where it stays with probability 1. We equip each state of the form $q_2 = (l_2, \nu_2)$ with an exponential distribution of parameter $\lambda_2$ over the delays. Then it can be shown that the probability to reach $B_1$ in $\mathcal{A}_1$ corresponds to the probability to reach $(B_1, B_2)$ in $\mathcal{A}_1 \parallel \mathcal{A}_2$ iff $\ln(\lambda_1) - \ln(\lambda_2) = \lambda_1 - \lambda_2$ which is not true in general (in particular with $\lambda_1 = 1$ and $\lambda_2 = 2$).

We assume that the reader is familiar with exponential distributions. The product $\mathcal{A}_1 \parallel \mathcal{A}_2$ is depicted below. From the initial state $q = (q_1, q_2) = ((l_1, 0), (l_2, 0))$, it should be noted that both clocks will always have the same value since we never reset any clock to 0. Then, following Definition 3, one can see that

$$\mu_{((l_1,\nu),(l_2,\nu))} = \begin{cases} \mathrm{Exp}(\lambda_1 + \lambda_2) & \text{if } \nu < 1, \\ \mathrm{Exp}(2 \cdot \lambda_2) & \text{otherwise} \end{cases}$$

where $\mathrm{Exp}(\lambda)$ denotes the exponential distribution of parameter $\lambda$, and that for every $t \geq 0$,

$$w^{(1)}_{((l_1,\nu),(l_2,\nu))}(t) = \begin{cases} \frac{\lambda_1}{\lambda_1+\lambda_2} & \text{if } \nu < 1, \\ \frac{1}{2} & \text{otherwise,} \end{cases} \quad \text{and} \quad w^{(2)}_{((l_1,\nu),(l_2,\nu))}(t) = \begin{cases} \frac{\lambda_2}{\lambda_1+\lambda_2} & \text{if } \nu < 1, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$




$\mathcal{A}_1 \notin$ CSTA

In states of the form $((B_1, \nu), (l_2, \nu))$ and $(((G_1, \nu), (l_2, \nu))$ we keep the distributions of $(B_1, \nu)$ and $(l_2, \nu)$, while in $((l_1, \nu), (B_2, \nu))$ we keep the distributions of $(l_1, \nu)$.

Now, one can observe that the set of runs in $\mathcal{A}_1$ starting in $q_1$ that reach $B_1$ after 2 time units is given by $\mathrm{Cyl}(\pi(q_1, e_1))$, while the set of runs in $\mathcal{A}_1 \parallel \mathcal{A}_2$ starting in $q$ that reach $B_1$ is given by $\mathrm{Cyl}(\pi(q, e_1)) \cup \mathrm{Cyl}(\pi(q, e_3, e_1))$. Correspondingly to our interleaving semantics, we would like that

$$\mathbb{P}_{\mathcal{A}_1 \parallel \mathcal{A}_2}\big(\mathrm{Cyl}(\pi(q, e_1)) \cup \mathrm{Cyl}(\pi(q, e_3, e_1))\big) = \mathbb{P}_{\mathcal{A}_1}\big(\mathrm{Cyl}(\pi(q_1, e_1))\big). \tag{C.8}$$

It can easily be established that

$$\mathbb{P}_{\mathcal{A}_1}\big(\mathrm{Cyl}(\pi(q_1, e_1))\big) = e^{-2\lambda_1}. \tag{C.9}$$

We can compute

$$\begin{aligned}
&\mathbb{P}_{\mathcal{A}_1 \parallel \mathcal{A}_2}\big(\mathrm{Cyl}(\pi(q, e_1)) \cup \mathrm{Cyl}(\pi(q, e_3, e_1))\big) \\
&\quad = \mathbb{P}_{\mathcal{A}_1 \parallel \mathcal{A}_2}\big(\mathrm{Cyl}(\pi(q, e_1))\big) + \mathbb{P}_{\mathcal{A}_1 \parallel \mathcal{A}_2}\big(\mathrm{Cyl}(\pi(q, e_3, e_1))\big) \\
&\quad = \int_2^\infty \frac{\lambda_1}{\lambda_1 + \lambda_2} \cdot (\lambda_1 + \lambda_2) \cdot e^{-(\lambda_1 + \lambda_2)t} \, \mathrm{d}t \\
&\quad + \int_{t_1 = 0}^{1} \frac{\lambda_2}{\lambda_1 + \lambda_2} \cdot (\lambda_1 + \lambda_2) \cdot e^{-(\lambda_1 + \lambda_2)t_1} \int_{t_2 = 2 - t_1}^{\infty} \lambda_1 \cdot e^{-\lambda_1 t_2} \, \mathrm{d}t_2 \, \mathrm{d}t_1 \\
&\quad + \int_{t_1 = 1}^{2} \frac{\lambda_2}{\lambda_1 + \lambda_2} \cdot (\lambda_1 + \lambda_2) \cdot e^{-(\lambda_1 + \lambda_2)t_1} \int_{t_2 = 2 - t_1}^{\infty} \lambda_2 \cdot e^{-\lambda_2 t_2} \, \mathrm{d}t_2 \, \mathrm{d}t_1 \\
&\quad + \int_2^\infty \frac{\lambda_2}{\lambda_1 + \lambda_2} \cdot (\lambda_1 + \lambda_2) \cdot e^{-(\lambda_1 + \lambda_2)t} \, \mathrm{d}t \\
&\quad = e^{-2(\lambda_1 + \lambda_2)} + e^{-2\lambda_1} \cdot (1 - e^{-\lambda_2}) + \frac{\lambda_2}{\lambda_1} \cdot e^{-2\lambda_2} \cdot (e^{-\lambda_1} - e^{-2\lambda_1}).
\end{aligned} \tag{C.10}$$

Hence, from (C.9) and (C.10), we have (C.8) iff

$$
\begin{aligned}
&e^{-2(\lambda_1+\lambda_2)} + e^{-2\lambda_1} - e^{-2\lambda_1-\lambda_2} + \frac{\lambda_2}{\lambda_1} \cdot e^{-2\lambda_2} \cdot (e^{-\lambda_1} - e^{-2\lambda_1}) = e^{-2\lambda_1} \\
\Leftrightarrow &\frac{\lambda_2}{\lambda_1} \cdot e^{-2\lambda_2} \cdot (e^{-\lambda_1} - e^{-2\lambda_1}) = e^{-2\lambda_1-\lambda_2} - e^{-2(\lambda_1+\lambda_2)} \\
\Leftrightarrow &\frac{\lambda_2}{\lambda_1} \cdot e^{-2\lambda_2} \cdot (e^{-\lambda_1} - e^{-2\lambda_1}) = e^{-2\lambda_2} \cdot (e^{-2\lambda_1+\lambda_2} - e^{-2\lambda_1}) \\
\Leftrightarrow &\frac{\lambda_2}{\lambda_1} \cdot e^{-\lambda_1} = e^{-2\lambda_1+\lambda_2} \\
\Leftrightarrow &\lambda_2 e^{-\lambda_2} = \lambda_1 e^{-\lambda_1} \\
\Leftrightarrow &\ln(\lambda_1) - \ln(\lambda_2) = \lambda_1 - \lambda_2.
\end{aligned}
$$

**Details for Example 3**

In order to illustrate the notion of composition, we composed two independent copies of the STA modelling the IPv4 Zeroconf protocol (see Example 1). Part of the composed STA is depicted in Figure 3 (repeated below).

We consider two STA $\mathcal{A}_1$ and $\mathcal{A}_2$, modelling each the IPv4 Zeroconf protocol and that run completely independently. We explain how we compute in $\mathcal{A}_1 \parallel \mathcal{A}_2$, the probability distributions over delays and edges in location $(\mathrm{IP}_1, \mathrm{IP}_2)$. We consider a state of the form $q = (q_1, q_2)$ with $q_1 = (\mathrm{IP}_1, \nu_1)$ and $q_2 = (\mathrm{IP}_2, \nu_2)$. We thus have that $I(q_1) = \mathbb{R}_+ \setminus \{T - \nu_1\}$ and $I(q_2) = \mathbb{R}_+ \setminus \{T - \nu_2\}$. We suppose that $q_1$ (resp. $q_2$) is equipped with $\mathrm{Exp}(\mu_1)$ (resp. $\mathrm{Exp}(\mu_2)$) for the probability over the delays. Then the law of the minimum between these two probabilities is an exponential distribution of parameter $\mu_1 + \mu_2$, i.e. $\mathrm{Exp}(\mu_1 + \mu_2)$. Now one can see that for every $t \geq 0$, from $q + t$, there are two enabled edges: $e_1$ which is the only edge enabled from $q_1 + t$ in $\mathcal{A}_1$ and $e_2$ which is the only edge enabled from $q_2 + t$ in $\mathcal{A}_2$. Hence, one can compute

$$
p_{q+t}(e_1) = w_q^1(t) = \frac{\mu_1}{\mu_1 + \mu_2} \quad \text{and} \quad p_{q+t}(e_2) = w_q^2(t) = \frac{\mu_2}{\mu_1 + \mu_2}.
$$

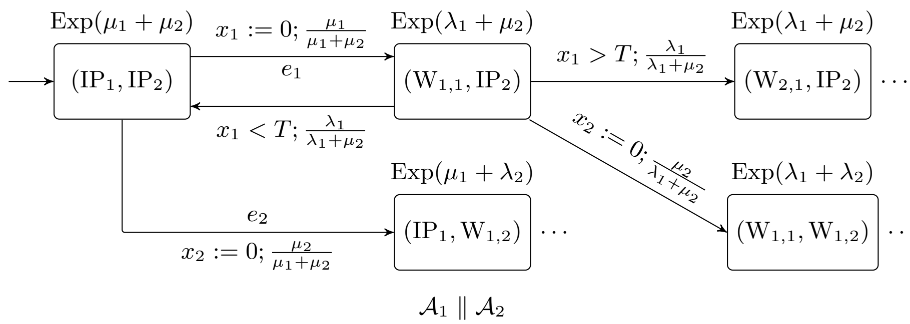
The rest of the automaton can be completed in a similar way.

**Details for Theorem 2**

We recall the definition of the class CSTA*: we say that $\mathcal{A} \in \mathrm{CSTA}^*$ if $\mathcal{A} \in \mathrm{CSTA}$ and

(C) $\mathcal{A}$ is almost-surely non -Zeno.

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two composable STA. We write $\iota_1$ (resp. $\iota_2$) for the natural projection of $\mathrm{Runs}(\mathcal{A}_1 \parallel \mathcal{A}_2, q)$ onto $\mathrm{Runs}(\mathcal{A}_1, q_1)$ (resp. $\mathrm{Runs}(\mathcal{A}_2, q_2)$) for each state $q = (q_1, q_2) \in Q_1 \times Q_2$: we inductively define $\iota_1(q) = q_1$, $\iota_1(q \xrightarrow{t_1, e_1} q^{(1)}) = q_1 \xrightarrow{t_1, e_1} q_1^{(1)}$ if $e_1 \in E_1$; $\iota_1(q \xrightarrow{t_1, e_1} q^{(1)} \xrightarrow{t_2, e_2} q^{(2)}) = q_1 \xrightarrow{t_1 + t_2, e_2} q_1^{(2)}$ if $e_1 \in E_2$ and $e_2 \in E_1$, ...




The product of two STA modelling the IPv4 Zeroconf for $N = 3$

**Theorem 2.** *Let $\mathcal{A}_1, \mathcal{A}_2 \in \mathrm{CSTA}^*$ be composable. Then $\mathcal{A}_1 \parallel \mathcal{A}_2 \in \mathrm{CSTA}^*$. Moreover, for every state $q = (q_1, q_2)$ of $\mathcal{A}_1 \parallel \mathcal{A}_2$, for every properties $\varphi_1$ measurable in $\mathcal{A}_1$ from $q_1$ and $\varphi_2$ measurable in $\mathcal{A}_2$ from $q_2$, we have*

$$\mathbb{P}_{\mathcal{A}_1\parallel\mathcal{A}_2}(\{q \models \widetilde{\varphi}_1\} \cap \{q \models \widetilde{\varphi}_2\}) = \mathbb{P}_{\mathcal{A}_1}(\{q_1 \models \varphi_1\}) \cdot \mathbb{P}_{\mathcal{A}_2}(\{q_2 \models \varphi_2\}). \quad (2)$$

*Proof.* Given $\mathcal{A}_1$ and $\mathcal{A}_2 \in \mathrm{CSTA}^*$, thanks to Lemma 1, in order to get that $\mathcal{A}_1 \parallel \mathcal{A}_2 \in \mathrm{CSTA}^*$, it suffices to prove that $\mathcal{A}_1 \parallel \mathcal{A}_2$ is almost-surely non-Zeno. This will be ensured by (2). We thus first tackle the proof of (2).

Let $q = (q_1, q_2) = ((l_1, \nu_1), (l_2, \nu_2))$ be a state of $\mathcal{A}_1 \parallel \mathcal{A}_2$. The important first step to prove (2) consists in showing that, given an edge $e_1$ of $\mathcal{A}_1$, the probability in $\mathcal{A}_1 \parallel \mathcal{A}_2$ that $e_1$ is the first edge performed from $q = (q_1, q_2)$ in a given set of delays $\mathcal{C}$ corresponds to the probability in $\mathcal{A}_1$ that $e_1$ is the first edge performed from $q_1$ in the same set of delays $\mathcal{C}$, that is for every Borel set $\mathcal{C}$ of $\mathbb{R}_+$,

$$\mathbb{P}_{\mathcal{A}_1\parallel\mathcal{A}_2}(\mathrm{Cyl}(\pi_{\mathcal{C}^*}(q, \mathcal{A}_2^*, e_1))) = \mathbb{P}_{\mathcal{A}_1}(\mathrm{Cyl}(\pi_{\mathcal{C}}(q_1, e_1))) \quad \text{(C.11)}$$

where $\mathrm{Cyl}(\pi_{\mathcal{C}^*}(q, \mathcal{A}_2^*, e_1)) = \iota_1^{-1}(\mathrm{Cyl}(\pi_{\mathcal{C}}(q_1, e_1)))$. Formally, we have that

$$\mathrm{Cyl}(\pi_{\mathcal{C}^*}(q, \mathcal{A}_2^*, e_1)) = \bigcup_{n\in\mathbb{N}} \bigcup_{(f_1,\ldots,f_n)\in E_2^n} \mathrm{Cyl}(\pi_{\mathcal{C}_n}(q, f_1, \ldots, f_n, e_1))$$

where $\mathcal{C}_n = \{(\tau_1, \ldots, \tau_{n+1}) \in \mathbb{R}_+^{n+1} \mid \tau_1 + \ldots + \tau_{n+1} \in \mathcal{C}\}$ for every $n \in \mathbb{N}$, which is a countable union of disjoint cylinders. In order to show (C.11), hypothesis (B) is crucial. Indeed, if for instance $e_1 \in E_1$ and $e_2 \in E_2$, then the projection of $q \xrightarrow{\tau_1, e_2} \cdot \xrightarrow{\tau_2, e_1} q'$ in $\mathcal{A}_1$ is $q_1 \xrightarrow{\tau_1} q_1 + \tau_1 \xrightarrow{\tau_2, e_1} q_1'$ which is equivalent to $q_1 \xrightarrow{\tau_1+\tau_2, e_1} q_1'$: the first movement $q \xrightarrow{\tau_1, e_2} \cdot$ is in $\mathcal{A}_2$ and has no impact over $\mathcal{A}_1$, except the elapse of $\tau_1$ time units. Hypothesis (B) ensures that the probability in $\mathcal{A}_1$ to leave $q_1$ after $\tau_1 + \tau_2$ time units knowing that we leave it after at least $\tau_1$ time units coincides with the probability to leave $q_1 + \tau_1$ after $\tau_2$ time units. It is formalized in the next proposition.

**Proposition 4.** *Assuming the above notations, for every $e_1 \in E_1$ and for every Borel set $\mathcal{C}$ of $\mathbb{R}_+$, we have*

$$\mathbb{P}_{\mathcal{A}_1 \| \mathcal{A}_2}(\mathrm{Cyl}(\pi_{\mathcal{C}^*}(q, \mathcal{A}_2^*, e_1))) = \mathbb{P}_{\mathcal{A}_1}(\mathrm{Cyl}(\pi_{\mathcal{C}}(q_1, e_1))). \tag{C.11}$$

*Proof.* We first assume that $\mathcal{C} = \mathbb{R}_+$. We have to show that for every $e_1 \in E_1$,

$$\sum_{n \geq 0} p_n(q) = \mathbb{P}_{\mathcal{A}_1}(\mathrm{Cyl}(\pi(q_1, e_1))) \tag{C.12}$$

where $p_n(q)$ is the probability of the set of terminal runs in $\mathcal{A}_1 \parallel \mathcal{A}_2$ that start in $q$ and that first perform $n$ switch-transitions in $E_2$ and then choose $e_1$ as the first edge of $E_1$, i.e.

$$p_n(q) = \mathbb{P}_{\mathcal{A}_1 \| \mathcal{A}_2}\left( \bigcup_{(f_1, \ldots, f_n) \in E_2^n} \mathrm{Cyl}(\pi(q, f_1, \ldots, f_n, e_1)) \right).$$

In order to prove (C.12), we first show that for each $n \geq 0$,

$$\sum_{i=0}^{n-1} p_i(q) + p'_n(q) = \mathbb{P}_{\mathcal{A}_1}(\mathrm{Cyl}(\pi(q_1, e_1))),$$

where $p'_n(q)$ is the probability of the set of terminal runs in $\mathcal{A}_1 \parallel \mathcal{A}_2$ that start in $q$ and that first perform $n$ switch-transitions in $E_2$ and then choose $e_1$ as the first edge of $E_1$, knowing that the $n+1$th transition is won with probability 1 by $\mathcal{A}_1$. This is proved in Lemma 10. The key point of this result lies in the fact that $p_n(q)$ corresponds to the probability that $\mathcal{A}_1$ chooses first $e_1$, that $\mathcal{A}_2$ perform its $n$ transitions before $\mathcal{A}_1$ performs $e_1$, and that its $n+1$th transition must be taken after $e_1$, while $p'_n(q)$ corresponds to the probability that $\mathcal{A}_1$ chooses first $e_1$ and that $\mathcal{A}_2$ perform its $n$ transitions before $\mathcal{A}_1$. It is formalized in Lemma 8 and 9. These lemmas will lead to the fact that $p_n(q) + p'_{n+1}(q) = p'_n(q)$ for every $n \geq 0$. We then show that $p'_n(q) \underset{n \to +\infty}{\longrightarrow} 0$ which will conclude the proof of Proposition 4.

In the sequel, we keep the same notations as before for the density and cumulative functions of the probability measures $\mu_{q_i}$, and we refer to Definition 3 for the probability measures considered in the automaton $\mathcal{A}_1 \parallel \mathcal{A}_2$. Given a state $q$ of $\mathcal{A}_1 \parallel \mathcal{A}_2$, given an edge $e \in E_1$ and an edge $f \in E_2$, we abusively write $I(q, e)$ for $I(q_1, e)$ and $I(q, f)$ for $I(q_2, f)$. Let us recall that given $f_1, \ldots, f_n$ we write $q_{t_1 \ldots t_n}$ for the state such that

$$q \xrightarrow{t_1, f_1} \cdot \xrightarrow{t_2, f_2} \ldots \xrightarrow{t_n, f_n} q_{t_1 \ldots t_n}.$$

Let us notice that if $f_1, \ldots, f_n$ are all in $E_2$ then the projection of $q_{t_1 \ldots t_n}$ in $\mathcal{A}_1$ is given by $q_1 + t_1 + \ldots + t_n$.

**Lemma 8.** *Assuming the above notations, for every $n \geq 0$, we have*

$$p_n(q) = \sum_{(f_1,\ldots,f_n)\in E_2^n} \int_{t\in I(q,e_1)} f_{q_1}(t) p_{q_1+t}(e_1) \int_{t_1=0}^{t} f_{q_2}(t_1) p_{q_2+t_1}(f_1) \mathbb{1}_{I(q,f_1)}(t_1)$$
$$\int_{t_2=0}^{t-t_1} f_{q_{t_1},2}(t_2) p^{(2)}_{q_{t_1}+t_2}(f_2) \mathbb{1}_{I(q_{t_1},f_2)}(t_2) \ldots$$
$$\int_{t_n=0}^{t-t_1\ldots-t_{n-1}} f_{q_{t_1\ldots t_{n-1}},2}(t_n) p^{(2)}_{q_{t_1\ldots t_{n-1}}+t_n}(f_n) \mathbb{1}_{I(q_{t_1\ldots t_{n-1}},f_n)}(t_n)$$
$$\big(1 - F_{q_{t_1\ldots t_n},2}(t - t_1 - \ldots - t_n)\big) \,\mathrm{d}t_n \ldots \mathrm{d}t_2 \,\mathrm{d}t_1 \,\mathrm{d}t. \tag{C.13}$$

*Proof.* We prove the lemma by induction over $n$. If $n = 0$, then we have from Definition 3

$$\begin{aligned}
p_0(q) &= \mathbb{P}_{\mathcal{A}_1\|\mathcal{A}_2}(\mathrm{Cyl}(\pi(q, e_1))) \\
&= \int_{t\in I(q,e_1)} f_q(t) p_{q+t}(e_1) \,\mathrm{d}t \\
&= \int_{t\in I(q,e_1)} f_q(t) w_q^1(t) p_{q_1+t}(e_1) \,\mathrm{d}t \\
&= \int_{t\in I(q,e_1)} f_q(t) \frac{f_{q_1}(t)(1 - F_{q_2}(t))}{f_q(t)} p_{q_1+t}(e_1) \,\mathrm{d}t \\
&= \int_{t\in I(q,e_1)} f_{q_1}(t) p_{q_1+t}(e_1)(1 - F_{q_2}(t)) \,\mathrm{d}t,
\end{aligned}$$

and thus equality (C.13) is satisfied for $n = 0$ , for every state $q$ of $\mathcal{A}_1 \parallel \mathcal{A}_2$. Now, let $n \geq 0$ and let us assume that (C.13) is verified for every state $q$ of $\mathcal{A}_1 \parallel \mathcal{A}_2$ and for every $0 \leq k \leq n$. We now show that it is still the case for $k = n + 1$. Let $q$ be a state of $\mathcal{A}_1 \parallel \mathcal{A}_2$, we have

$$p_{n+1}(q) = \mathbb{P}_{\mathcal{A}_1\|\mathcal{A}_2}\left( \bigcup_{(f_1,\ldots,f_{n+1})\in E_2^{n+1}} \mathrm{Cyl}(\pi(q, f_1, \ldots, f_{n+1}, e_1)) \right)$$

and thus, since $\mathrm{Cyl}(\pi(q, f_1, \ldots, f_{n+1}, e_1)) \cap \mathrm{Cyl}(\pi(q, f_1', \ldots, f_{n+1}', e_1)) = \emptyset$ whenever $(f_1, \ldots, f_{n+1}) \neq (f_1', \ldots, f_{n+1}')$, we have from Definition 3

$$\begin{aligned}
&p_{n+1}(q) \\
&\quad= \sum_{(f_1,\ldots,f_{n+1})\in E_2^{n+1}} \mathbb{P}_{\mathcal{A}_1\|\mathcal{A}_2}(\mathrm{Cyl}(\pi(q, f_1, \ldots, f_{n+1}, e_1))) \\
&\quad= \sum_{(f_1,\ldots,f_{n+1})\in E_2^{n+1}} \int_{t_1\in I(q,f_1)} f_{q_2}(t_1)(1 - F_{q_1}(t_1)) p_{q_2+t_1}(f_1) \\
&\qquad\qquad \mathbb{P}_{\mathcal{A}_1\|\mathcal{A}_2}(\mathrm{Cyl}(\pi(q_{t_1}, f_2, \ldots, f_{n+1}, e_1))) \,\mathrm{d}t_1.
\end{aligned}$$

Now, since the value $f_{q_2}(t_1)(1-F_{q_1}(t_1))p_{q_2+t_1}(f_1)$ only depends on $f_1$ and since $E_2$ is a finite set, we have

$$\begin{aligned}
p_{n+1}(q) &= \sum_{f_1\in E_2}\int_{t_1\in I(q,f_1)} f_{q_2}(t_1)(1-F_{q_1}(t_1))p_{q_2+t_1}(f_1)\cdot \\
&\qquad \left(\sum_{(f_2,\ldots,f_{n+1})\in E_2^n} \mathbb{P}_{\mathcal{A}_1\|\mathcal{A}_2}(\mathrm{Cyl}(\pi(q_{t_1},f_2,\ldots,f_{n+1},e_1)))\right)\mathrm{d}t_1 \\
&= \sum_{f_1\in E_2}\int_{t_1\in I(q,f_1)} f_{q_2}(t_1)(1-F_{q_1}(t_1))p_{q_2+t_1}(f_1)p_n(q_{t_1})\,\mathrm{d}t_1.
\end{aligned}$$

Now from the hypothesis of induction we can compute $p_n(q_{t_1})$ with (C.13), and thus

$$\begin{aligned}
p_{n+1}(q) = &\sum_{(f_1,\ldots,f_{n+1})\in E_2^{n+1}}\int_{t_1\in I(q,f_1)} f_{q_2}(t_1)(1-F_{q_1}(t_1))p_{q_2+t_1}(f_1) \\
&\int_{u\in I(q+t_1,e_1)} f_{q_1+t_1}(u)p^{(1)}_{q+t_1+u}(e_1)\int_{t_2=0}^{u} f_{q_{t_1},2}(t_2)p^{(2)}_{q_{t_1}+t_2}(f_2)\mathbb{1}_{I(q_{t_1},f_2)}(t_2) \\
&\int_{t_3=0}^{u-t_2} f_{q_{t_1t_2},2}(t_3)p^{(2)}_{q_{t_1t_2}+t_3}(f_3)\mathbb{1}_{I(q_{t_1t_2},f_3)}(t_3)\ldots \\
&\int_{t_{n+1}=0}^{u-t_2\ldots-t_n} f_{q_{t_1\ldots t_n},2}(t_{n+1})p^{(2)}_{q_{t_1\ldots t_n}+t_{n+1}}(f_{n+1})\mathbb{1}_{I(q_{t_1\ldots t_n},f_{n+1})}(t_{n+1}) \\
&\big(1-F_{q_{t_1\ldots t_{n+1}},2}(u-t_2-\ldots-t_{n+1})\big)\,\mathrm{d}t_{n+1}\ldots\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}u\,\mathrm{d}t_1
\end{aligned}$$

since the projection of $q_{t_1}$ in $\mathcal{A}_1$ is $q+t_1$ (as $q_{t_1}$ is such that $q\xrightarrow{t_1,f_1}q_{t_1}$ and $f_1\in E_2$) and since $E_2$ is a finite set. Under the hypotheses over $\mathcal{A}_1$ and $\mathcal{A}_2$, we know that $f_{q_1+t_1}(u)(1-F_{q_1}(t_1)) = f_{q_1}(u+t_1)$. Now, if we let $t=u+t_1$, then $u=t-t_1$ and we have that for every $t_1\geq 0$,

$$\begin{aligned}
u\in I(q+t_1,e_1) &\Leftrightarrow \nu_1+t_1+u\models g_{e_1} \\
&\Leftrightarrow \nu_1+t\models g_{e_1}\quad\text{and}\quad t\geq t_1 \\
&\Leftrightarrow t\in I(q,e_1)\cap[t_1,+\infty[
\end{aligned}$$

where $g_{e_1}$ denotes the guard of edge $e_1$. From classical results of integration by substitution, we obtain that

$$p_{n+1}(q) = \sum_{(f_1,\dots,f_{n+1})\in E_2^{n+1}} \int_{t_1\in I(q,f_1)} f_{q_2}(t_1) p_{q_2+t_1}(f_1)$$
$$\int_{t\in I(q,e_1)} f_{q_1}(t) p_{q_1+t}(e_1) \mathbb{1}_{[t_1,+\infty[}(t) \int_{t_2=0}^{t-t_1} f_{q_{t_1},2}(t_2) p^{(2)}_{q_{t_1}+t_2}(f_2) \mathbb{1}_{I(q_{t_1},f_2)}(t_2)$$
$$\int_{t_3=0}^{t-t_1-t_2} f_{q_{t_1t_2},2}(t_3) p^{(2)}_{q_{t_1t_2}+t_3}(f_3) \mathbb{1}_{I(q_{t_1t_2},f_3)}(t_3) \dots$$
$$\int_{t_{n+1}=0}^{t-t_1\dots-t_n} f_{q_{t_1\dots t_n},2}(t_{n+1}) p^{(2)}_{q_{t_1\dots t_n}+t_{n+1}}(f_{n+1}) \mathbb{1}_{I(q_{t_1\dots t_n},f_{n+1})}(t_{n+1})$$
$$\big(1 - F_{q_{t_1\dots t_{n+1}},2}(t - t_1 - \dots - t_{n+1})\big) \,\mathrm{d}t_{n+1} \dots \mathrm{d}t_3 \,\mathrm{d}t_2 \,\mathrm{d}t \,\mathrm{d}t_1.$$

Now using the fact that $\mathbb{1}_{[t_1,+\infty[}(t) = \mathbb{1}_{[0,t]}(t_1)$ and using Fubini's theorem, we deduce that (C.13) is satisfied for $n+1$ which concludes the proof of the lemma. $\square$

**Lemma 9.** *Assuming the above notations, for every $n \geq 0$, we have*

$$p'_n(q) = \sum_{(f_1,\dots,f_n)\in E_2^n} \int_{t\in I(q,e_1)} f_{q_1}(t) p_{q_1+t}(e_1) \int_{t_1=0}^{t} f_{q_2}(t_1) p_{q_2+t_1}(f_1) \mathbb{1}_{I(q,f_1)}(t_1)$$
$$\int_{t_2=0}^{t-t_1} f_{q_{t_1},2}(t_2) p^{(2)}_{q_{t_1}+t_2}(f_2) \mathbb{1}_{I(q_{t_1},f_2)}(t_2) \dots$$
$$\int_{t_n=0}^{t-t_1\dots-t_{n-1}} f_{q_{t_1\dots t_{n-1}},2}(t_n) p^{(2)}_{q_{t_1\dots t_{n-1}}+t_n}(f_n) \mathbb{1}_{I(q_{t_1\dots t_{n-1}},f_n)}(t_n)$$
$$\mathrm{d}t_n \dots \mathrm{d}t_2 \,\mathrm{d}t_1 \,\mathrm{d}t. \tag{C.14}$$

*Proof.* We recall that, from the above notations, for every $n \geq 0$, $p'_n(q)$ is the probability of the set of terminal runs in $\mathcal{A}_1 \parallel \mathcal{A}_2$ that start in $q$ and that first perform $n$ switch-transitions in $E_2$ and then choose $e_1$ in the case where, after that $\mathcal{A}_2$ has performed $n$ transitions, then the $n+1$th transition is won with probability 1 by $\mathcal{A}_1$. In other words in the case where, for every run $\rho = q \xrightarrow{t_1,f_1} \cdot \xrightarrow{t_2,f_2} \dots \xrightarrow{t_n,f_n} q_{t_1\dots t_n}$ with $f_1,\dots,f_n \in E_2$, we have $w^1_{q_{t_1\dots t_n}}(t) = 1$ for every $t \in I(q_{t_1\dots t_n})$. Now, let $s$ be an arbitrary state of $\mathcal{A}_1 \parallel \mathcal{A}_2$. From Definition 3, for every $t \in I(s)$, we have

$$\begin{aligned} w^1_s(t) = 1 &\Leftrightarrow \frac{f_{s_1}(t)(1 - F_{s_2}(t))}{f_s(t)} = 1 \\ &\Leftrightarrow f_{s_1}(t)(1 - F_{s_2}(t)) = f_{s_1}(t)(1 - F_{s_2}(t)) + f_{s_2}(t)(1 - F_{s_1}(t)) \\ &\Leftrightarrow f_{s_2}(t)(1 - F_{s_1}(t)) = 0. \end{aligned}$$

Thus, if for every $t \in I(s)$, $w^1_s(t) = 1$ then for every $t \in I(s)$, $f_{s_2}(t)(1 - F_{s_1}(t)) = 0$. We can then prove that $f_s(t) = f_{s_1}(t)$ almost-surely. Let us prove (C.14) when

$n = 0$. Let $q$ be a state of $\mathcal{A}_1 \parallel \mathcal{A}_2$, we have that $w^1_q(t) = 1$ for every $t \in I(q)$. And thus,

$$\begin{aligned} p'_0(q) &= \int_{t \in I(q,e_1)} f_q(t) w^1_q(t) p_{q_1+t}(e_1)\,\mathrm{d}t \\ &= \int_{t \in I(q,e_1)} f_{q_1}(t) p_{q_1+t}(e_1)\,\mathrm{d}t. \end{aligned}$$

Hence, (C.14) holds for every state $q$ when $n = 0$. We can then prove by induction over $n$ that if for every $0 \le k \le n$ with $n \ge 0$, (C.14) is satisfied for every state $q$, then it is still satisfied for $k = n+1$, for every state $q$. It uses similar arguments as in proof of Lemma 8. This concludes the proof. □

**Lemma 10.** *Assuming the above notations, for every $n \ge 0$, we have*

$$\sum_{i=0}^{n-1} p_i(q) + p'_n(q) = \mathbb{P}_{\mathcal{A}_1}(\mathrm{Cyl}(\pi(q_1, e_1))).$$

*Proof.* If $n = 0$, we have from Lemma 9 that $p'_0(q) = \mathbb{P}_{\mathcal{A}_1}(\mathrm{Cyl}(\pi(q_1, e_1)))$. Now, let us assume that for every $0 \le k \le n$ with $n \ge 0$, we have

$$\sum_{i=0}^{k-1} p_i(q) + p'_k(q) = \mathbb{P}_{\mathcal{A}_1}(\mathrm{Cyl}(\pi(q_1, e_1))),$$

and let us prove that it is still the case when $k = n+1$. First, let us compute $p'_{n+1}(q)$. From Lemma 9, we have

$$\begin{aligned} &p'_{n+1}(q) = \\ &\sum_{(f_1,\ldots,f_{n+1}) \in E_2^{n+1}} \int_{t \in I(q,e_1)} f_{q_1}(t) p_{q_1+t}(e_1) \int_{t_1=0}^{t} f_{q_2}(t_1) p_{q_2+t_1}(f_1) \mathbb{1}_{I(q,f_1)}(t_1) \\ &\qquad \int_{t_2=0}^{t-t_1} f_{q_{t_1},2}(t_2) p^{(2)}_{q_{t_1}+t_2}(f_2) \mathbb{1}_{I(q_{t_1},f_2)}(t_2) \ldots \\ &\qquad \int_{t_n=0}^{t-t_1\ldots-t_{n-1}} f_{q_{t_1\ldots t_{n-1}},2}(t_n) p^{(2)}_{q_{t_1\ldots t_{n-1}}+t_n}(f_n) \mathbb{1}_{I(q_{t_1\ldots t_{n-1}},f_n)}(t_n) \\ &\qquad \int_{t_{n+1}=0}^{t-t_1\ldots-t_n} f_{q_{t_1\ldots t_n},2}(t_{n+1}) p^{(2)}_{q_{t_1\ldots t_n}+t_{n+1}}(f_{n+1}) \mathbb{1}_{I(q_{t_1\ldots t_n},f_{n+1})}(t_{n+1}) \\ &\qquad\quad \mathrm{d}t_{n+1}\,\mathrm{d}t_n \ldots \mathrm{d}t_2\,\mathrm{d}t_1\,\mathrm{d}t. \end{aligned}$$

Now, one can observe that only $p^{(2)}_{q_{t_1\dots t_n}+t_{n+1}}(f_{n+1})\mathbb{1}_{I(q_{t_1\dots t_n},f_{n+1})}(t_{n+1})$ depends on $f_{n+1}$ in the last equality. Thus, since $E_2$ is finite, we have

$$\begin{aligned}
&p'_{n+1}(q) = \\
&\quad\sum_{(f_1,\dots,f_n)\in E_2^n} \int_{t\in I(q,e_1)} f_{q_1}(t)p_{q_1+t}(e_1)\int_{t_1=0}^{t} f_{q_2}(t_1)p_{q_2+t_1}(f_1)\mathbb{1}_{I(q,f_1)}(t_1)\\
&\qquad\int_{t_2=0}^{t-t_1} f_{q_{t_1},2}(t_2)p^{(2)}_{q_{t_1}+t_2}(f_2)\mathbb{1}_{I(q_{t_1},f_2)}(t_2)\dots\\
&\qquad\int_{t_n=0}^{t-t_1\dots-t_{n-1}} f_{q_{t_1\dots t_{n-1}},2}(t_n)p^{(2)}_{q_{t_1\dots t_{n-1}}+t_n}(f_n)\mathbb{1}_{I(q_{t_1\dots t_{n-1}},f_n)}(t_n)\\
&\quad\sum_{f_{n+1}\in E_2}\int_{t_{n+1}=0}^{t-t_1\dots-t_n} f_{q_{t_1\dots t_n},2}(t_{n+1})p^{(2)}_{q_{t_1\dots t_n}+t_{n+1}}(f_{n+1})\mathbb{1}_{I(q_{t_1\dots t_n},f_{n+1})}(t_{n+1})\\
&\qquad\quad \mathrm{d}t_{n+1}\,\mathrm{d}t_n\dots\mathrm{d}t_2\,\mathrm{d}t_1\,\mathrm{d}t.
\end{aligned}$$

Now, we have that

$$\begin{aligned}
&\sum_{f_{n+1}\in E_2}\int_{t_{n+1}=0}^{t-t_1\dots-t_n} f_{q_{t_1\dots t_n},2}(t_{n+1})p^{(2)}_{q_{t_1\dots t_n}+t_{n+1}}(f_{n+1})\mathbb{1}_{I(q_{t_1\dots t_n},f_{n+1})}(t_{n+1})\\
&\quad=\int_{t_{n+1}=0}^{t-t_1\dots-t_n} f_{q_{t_1\dots t_n},2}(t_{n+1})\\
&\qquad\sum_{f_{n+1}\in E_2}\left(p^{(2)}_{q_{t_1\dots t_n}+t_{n+1}}(f_{n+1})\mathbb{1}_{I(q_{t_1\dots t_n},f_{n+1})}(t_{n+1})\right)\mathrm{d}t_{n+1}\\
&\quad=\int_{t_{n+1}=0}^{t-t_1\dots-t_n} f_{q_{t_1\dots t_n},2}(t_{n+1})\mathbb{1}_{I(q_{t_1\dots t_n})}(t_{n+1})\\
&\quad=F_{q_{t_1\dots t_n},2}(t-t_1-\dots-t_n)
\end{aligned}$$

since $p^{(2)}_{q_{t_1\dots t_n}+t_{n+1}}$ is a probability measure over the enabled edges in $q_{t_1\dots t_n}+t_{n+1}$ when this set is not empty (otherwise, we assume that $p^{(2)}_{q_{t_1\dots t_n}+t_{n+1}}$ is a function that assigns 0 to each edge of $E_2$). We deduce thus that

$$\begin{aligned}
p'_{n+1}(q) = &\sum_{(f_1,\dots,f_n)\in E_2^n} \int_{t\in I(q,e_1)} f_{q_1}(t)p_{q_1+t}(e_1)\int_{t_1=0}^{t} f_{q_2}(t_1)p_{q_2+t_1}(f_1)\mathbb{1}_{I(q,f_1)}(t_1)\\
&\qquad\int_{t_2=0}^{t-t_1} f_{q_{t_1},2}(t_2)p^{(2)}_{q_{t_1}+t_2}(f_2)\mathbb{1}_{I(q_{t_1},f_2)}(t_2)\dots\\
&\qquad\int_{t_n=0}^{t-t_1\dots-t_{n-1}} f_{q_{t_1\dots t_{n-1}},2}(t_n)p^{(2)}_{q_{t_1\dots t_{n-1}}+t_n}(f_n)\mathbb{1}_{I(q_{t_1\dots t_{n-1}},f_n)}(t_n)\\
&\qquad\quad F_{q_{t_1\dots t_n},2}(t-t_1-\dots-t_n)\,\mathrm{d}t_n\dots\mathrm{d}t_2\,\mathrm{d}t_1\,\mathrm{d}t.
\end{aligned}$$

From this last equality and Lemmas 8 and 9, we can thus easily see that $p_n(q) + p'_{n+1}(q) = p'_n(q)$. From this last equality and the hypothesis of induction, we have

$$\begin{aligned}\sum_{i=0}^{n} p_i(q) + p'_{n+1}(q) &= \sum_{i=0}^{n-1} p_i(q) + p_n(q) + p'_{n+1}(q) \\ &= \sum_{i=0}^{n-1} p_i(q) + p'_n(q) \\ &= \mathbb{P}_{\mathcal{A}_1}(\mathrm{Cyl}(\pi(q_1, e_1)))\end{aligned}$$

which concludes the proof. □

**Lemma 11.** *Assuming the above notations, we have that* $p'_n(q) \underset{n\to+\infty}{\longrightarrow} 0$.

*Proof.* Since $E_2$ is finite we have that

$$\begin{aligned}p'_n(q) = & \int_{t\in I(q,e_1)} f_{q_1}(t) p_{q_1+t}(e_1) \\ \sum_{(f_1,\dots,f_n)\in E_2^n} & \int_{t_1=0}^{t} f_{q_2}(t_1) p_{q_2+t_1}(f_1) \mathbb{1}_{I(q,f_1)}(t_1) \\ & \int_{t_2=0}^{t-t_1} f_{q_{t_1},2}(t_2) p^{(2)}_{q_{t_1}+t_2}(f_2) \mathbb{1}_{I(q_{t_1},f_2)}(t_2) \dots \\ & \int_{t_n=0}^{t-t_1\dots-t_{n-1}} f_{q_{t_1\dots t_{n-1}},2}(t_n) p^{(2)}_{q_{t_1\dots t_{n-1}}+t_n}(f_n) \mathbb{1}_{I(q_{t_1\dots t_{n-1}},f_n)}(t_n) \\ & \quad \mathrm{d}t_n \dots \mathrm{d}t_2\, \mathrm{d}t_1\, \mathrm{d}t.\end{aligned}$$

We can then write $p'_n(q) = \int_{t\in I(q,e_1)} f(t) g_n(t)\, \mathrm{d}t$, where $f(t) = f_{q_1}(t) p_{q_1+t}(e_1)$ and

$$\begin{aligned}g_n(t) = & \sum_{(f_1,\dots,f_n)\in E_2^n} \int_{t_1=0}^{t} f_{q_2}(t_1) p_{q_2+t_1}(f_1) \mathbb{1}_{I(q,f_1)}(t_1) \\ & \int_{t_2=0}^{t-t_1} f_{q_{t_1},2}(t_2) p^{(2)}_{q_{t_1}+t_2}(f_2) \mathbb{1}_{I(q_{t_1},f_2)}(t_2) \dots \\ & \int_{t_n=0}^{t-t_1\dots-t_{n-1}} f_{q_{t_1\dots t_{n-1}},2}(t_n) p^{(2)}_{q_{t_1\dots t_{n-1}}+t_n}(f_n) \mathbb{1}_{I(q_{t_1\dots t_{n-1}},f_n)}(t_n)\, \mathrm{d}t_n \dots \mathrm{d}t_1.\end{aligned}$$

Under the hypotheses over $\mathcal{A}_1$ and $\mathcal{A}_2$, we know that $\mathcal{A}_2$ is almost-surely non-Zeno. Let $t \geq 0$ and let $M \in \mathbb{N}$ such that $t \leq M$. From Proposition 3 and from Remark 6, we have that

$$g_n(t) \leq \sum_{(f_1,\dots,f_n)\in E_2^n} \mathbb{P}_{\mathcal{A}_2}(\pi_{\mathcal{C}_{M,n}}(q, f_1, \dots, f_n)) \underset{n\to+\infty}{\longrightarrow} 0 \qquad \text{(C.15)}$$

and thus, $\lim_n g_n(t) = 0$ for every $t \geq 0$ as $g_n(t) \geq 0$ for every $t \geq 0$. Then, we have that $\lim_n f(t)g_n(t) = 0$ for every $t \geq 0$. Now, from inequality (C.15), we have that $g_n(t) \leq 1$ for every $t \geq 0$ and thus, $f(t)g_n(t) \leq f(t)$ for every $t \geq 0$. And since

$$\int_{t \in I(q,e_1)} f(t) = \int_{t \in I(q,e_1)} f_{q_1}(t) p_{q_1+t}(e_1)\,\mathrm{d}t \leq \int_{\mathbb{R}_+} f_{q_1}(t)\,\mathrm{d}t = 1,$$

we have, by dominated convergence, that

$$\lim_{n\to+\infty} p'_n(q) = \lim_{n\to+\infty} \int_{t \in I(q,e_1)} f(t) g_n(t)\,\mathrm{d}t = \int_{t \in I(q,e_1)} \lim_{n\to+\infty} (f(t)g_n(t))\,\mathrm{d}t = 0$$

which concludes the proof. □

We can now prove that (C.11) holds when $\mathcal{C} = \mathbb{R}_+$. We have

$$\begin{aligned}\mathbb{P}_{\mathcal{A}_1\|\mathcal{A}_2}(\mathrm{Cyl}(\pi(q, \mathcal{A}_2^*, e_1))) &= \mathbb{P}_{\mathcal{A}_1\|\mathcal{A}_2}\left(\bigcup_{n\geq 0} \bigcup_{(f_1,\ldots,f_n)\in E_2^n} \mathrm{Cyl}(\pi(f_1,\ldots,f_n,e_1))\right) \\ &= \sum_{n\geq 0} p_n(q) \qquad \text{(C.16)}\end{aligned}$$

from the definition of $p_n(q)$. Now, from Lemma 10, we have that for every $k \geq 0$,

$$\sum_{i=0}^{n} p_i(q) + p'_{n+1}(q) = \mathbb{P}_{\mathcal{A}_1}(\mathrm{Cyl}(\pi(q_1, e_1)))$$

and thus,

$$\lim_{n\to+\infty} \left(\sum_{i=0}^{n} p_i(q) + p'_{n+1}(q)\right) = \mathbb{P}_{\mathcal{A}_1}(\mathrm{Cyl}(\pi(q_1, e_1))).$$

Hence, since $\lim_n p'_{n+1}(q) = 0$ from Lemma 11, we have $\sum_{n\geq 0} p_n(q) = \mathbb{P}_{\mathcal{A}_1}(\mathrm{Cyl}(\pi(q_1, e_1)))$ and we deduce from (C.16) that

$$\mathbb{P}_{\mathcal{A}_1\|\mathcal{A}_2}(\mathrm{Cyl}(\pi(q, \mathcal{A}_2^*, e_1))) = \mathbb{P}_{\mathcal{A}_1}(\mathrm{Cyl}(\pi(q_1, e_1))).$$

Now, we would like to get (C.11) for every Borel set $\mathcal{C}$ of $\mathbb{R}_+$. Given a Borel set $\mathcal{C}$, a reasoning similar to the ones in Lemmas 8, 9, 10 and 11 can be applied to $p_{n,\mathcal{C}}(q)$ and $p'_{n,\mathcal{C}}(q)$ where

$$p_{n,\mathcal{C}}(q) = \mathbb{P}_{\mathcal{A}_1\|\mathcal{A}_2}\left(\bigcup_{(f_1,\ldots,f_n)} \mathrm{Cyl}(\pi_{\mathcal{C}_n}(q, f_1, \ldots, f_n, e_1))\right),$$

with $\mathcal{C}_n = \{(t_1, \ldots, t_{n+1}) \in \mathbb{R}_+^{n+1} \mid t_1 + \ldots + t_{n+1} \in \mathcal{C}\}$, is the probability of the set of terminal runs in $\mathcal{A}_1 \parallel \mathcal{A}_2$ that start in $q$ and that first perform $n$ switch-transitions in $E_2$ and then choose $e_1$ and that all these transitions are taken in a delay that is in $\mathcal{C}$, and $p'_n(q)$ is the probability of the same set in the case where, after that $\mathcal{A}_2$ has performed $n$ transitions, then the $n+1$th transition is won with probability 1 by $\mathcal{A}_1$. We can then prove, as in Lemmas 8 and 9, that for every state $q$, for every Borel set $\mathcal{C}$ and for every $n \geq 0$,

$$p_{n,\mathcal{C}}(q) = \sum_{(f_1,\ldots,f_n)\in E_2^n} \int_{t\in I(q,e_1)} f_{q_1}(t) p_{q_1+t}(e_1) \mathbb{1}_{\mathcal{C}}(t) \int_{t_1=0}^{t} f_{q_2}(t_1) p_{q_2+t_1}(f_1) \mathbb{1}_{I(q,f_1)}(t_1)$$
$$\int_{t_2=0}^{t-t_1} f_{q_{t_1},2}(t_2) p^{(2)}_{q_{t_1}+t_2}(f_2) \mathbb{1}_{I(q_{t_1},f_2)}(t_2) \ldots$$
$$\int_{t_n=0}^{t-t_1\ldots-t_{n-1}} f_{q_{t_1\ldots t_{n-1}},2}(t_n) p^{(2)}_{q_{t_1\ldots t_{n-1}}+t_n}(f_n) \mathbb{1}_{I(q_{t_1\ldots t_{n-1}},f_n)}(t_n)$$
$$\big(1 - F_{q_{t_1\ldots t_n},2}(t - t_1 - \ldots - t_n)\big)\, \mathrm{d}t_n \ldots \mathrm{d}t_2\, \mathrm{d}t_1\, \mathrm{d}t \qquad \text{(C.17)}$$

and

$$p'_{n,\mathcal{C}}(q) = \sum_{(f_1,\ldots,f_n)\in E_2^n} \int_{t\in I(q,e_1)} f_{q_1}(t) p_{q_1+t}(e_1) \mathbb{1}_{\mathcal{C}}(t)$$
$$\int_{t_1=0}^{t} f_{q_2}(t_1) p_{q_2+t_1}(f_1) \mathbb{1}_{I(q,f_1)}(t_1)$$
$$\int_{t_2=0}^{t-t_1} f_{q_{t_1},2}(t_2) p^{(2)}_{q_{t_1}+t_2}(f_2) \mathbb{1}_{I(q_{t_1},f_2)}(t_2) \ldots$$
$$\int_{t_n=0}^{t-t_1\ldots-t_{n-1}} f_{q_{t_1\ldots t_{n-1}},2}(t_n) p^{(2)}_{q_{t_1\ldots t_{n-1}}+t_n}(f_n) \mathbb{1}_{I(q_{t_1\ldots t_{n-1}},f_n)}(t_n)$$
$$\mathrm{d}t_n \ldots \mathrm{d}t_2\, \mathrm{d}t_1\, \mathrm{d}t. \qquad \text{(C.18)}$$

It can be proved by induction over $n$ as in the previous lemmas, by noticing that

$$p_{n+1,\mathcal{C}}(q) = \sum_{f_1\in E_2} \int_{t_1\in I(q,f_1)} f_{q_2}(t_1)(1 - F_{q_1}(t_1)) p_{q_2+t_1}(f_1) p_{n,\mathcal{C}_{t_1}}(q_{t_1})\, \mathrm{d}t_1 \qquad \text{(C.19)}$$

where $\mathcal{C}_{t_1}$ is a notation for $(\mathcal{C} - t_1) \cap \mathbb{R}_+$ and $(\mathcal{C} - t_1) = \{t - t_1 \mid t \in \mathcal{C}\}$, which is a Borel set. Indeed, we have that

$$\begin{aligned} p_{n+1,\mathcal{C}}(q) &= \sum_{(f_1,\ldots,f_{n+1})\in E_2^{n+1}} \mathbb{P}_{\mathcal{A}_1\|\mathcal{A}_2}(\mathrm{Cyl}(\pi_{\mathcal{C}_{n+1}}(q, f_1, \ldots, f_{n+1}, e_1))) \\ &= \sum_{(f_1,\ldots,f_{n+1})\in E_2^{n+1}} \int_{t_1\in I(q,f_1)} f_{q_2}(t_1)(1 - F_{q_1}(t_1)) p_{q_2+t_1}(f_1) \\ &\qquad \mathbb{P}_{\mathcal{A}_1\|\mathcal{A}_2}(\mathrm{Cyl}(\pi_{\mathcal{C}_{t_1,n}}(q_{t_1}, f_2, \ldots, f_{n+1}, e_1)))\, \mathrm{d}t_1 \end{aligned} \qquad \text{(C.20)}$$

where for every $t_1 \geq 0$, $\mathcal{C}_{t_1,n} = \{(t_2, \ldots, t_{n+2}) \in \mathbb{R}_+^{n+1} \mid t_1 + t_2 + \ldots + t_{n+2} \in \mathcal{C}\}$. Since we have that for every $t_1 \geq 0$, $(t_1, \ldots, t_{n+2}) \in \mathcal{C}_n$ if and only if

$(t_2, \ldots, t_{n+2}) \in \mathcal{C}_{t_1,n}$ where $\mathcal{C}_{t_1,n} = \{(t_2, \ldots, t_{n+2}) \in \mathbb{R}_+^{n+1} \mid (t_1, t_2, \ldots, t_{n+2}) \in \mathcal{C}\}$, we obtain

$$\mathbb{1}_{\mathcal{C}_n}((t_1, \ldots, t_{n+2})) = \mathbb{1}_{\mathcal{C}_{t_1,n}}((t_2, \ldots, t_{n+2})).$$

In our case, we can see that $\mathcal{C}_{t_1,n} = \{(t_2, \ldots, t_{n+2}) \in \mathbb{R}_+^{n+1} \mid t_2 + \ldots + t_{n+2} \in (\mathcal{C} - t_1) \cap \mathbb{R}_+\}$. We thus have (C.20) and as in Lemma 8, we deduce (C.19). If we assume by induction that for every state $q$ and for every Borel set $\mathcal{C}$, (C.17) holds, then we have

$$\begin{aligned}
p_{n+1,\mathcal{C}}(q) = \quad & \sum_{f_1 \in E_2} \int_{t_1 \in I(q,f_1)} f_{q_2}(t_1)(1 - F_{q_1}(t_1)) p_{q_2+t_1}(f_1) \\
\sum_{(f_2,\ldots,f_{n+1}) \in E_2^n} \quad & \int_{u \in I(q+t_1,e_1)} f_{q_1+t_1}(u) p^{(1)}_{q+t_1+u}(e_1) \mathbb{1}_{\mathcal{C}_{t_1}}(u) \\
& \int_{t_2=0}^{u} f_{q_{t_1},2}(t_2) p^{(2)}_{q_{t_1}+t_2}(f_2) \mathbb{1}_{I(q_{t_1},f_2)}(t_2) \\
& \int_{t_3=0}^{u-t_2} f_{q_{t_1 t_2},2}(t_3) p^{(2)}_{q_{t_1 t_2}+t_3}(f_3) \mathbb{1}_{I(q_{t_1 t_2},f_3)}(t_3) \ldots \\
& \int_{t_{n+1}=0}^{u-t_2 \ldots - t_n} f_{q_{t_1 \ldots t_n},2}(t_{n+1}) p^{(2)}_{q_{t_1 \ldots t_n}+t_{n+1}}(f_{n+1}) \mathbb{1}_{I(q_{t_1 \ldots t_n},f_{n+1})}(t_{n+1}) \\
& (1 - F_{q_{t_1 \ldots t_{n+1}},2}(u - t_2 - \ldots - t_{n+1}))\, \mathrm{d}t_{n+1} \ldots \mathrm{d}t_2\, \mathrm{d}u\, \mathrm{d}t_1.
\end{aligned}$$

Now, since $\mathcal{C}_{t_1} = (\mathcal{C} - t_1) \cap \mathbb{R}_+$ and since $u \geq 0$ for each $u \in I(q + t_1, e_1)$, we have that $\mathbb{1}_{\mathcal{C}_{t_1}}(u) = \mathbb{1}_{\mathcal{C}}(t_1 + u)$. Hence, by a substitution as in the proof of Lemma 8 and by Fubini's theorem, we obtain that (C.17) holds for $n + 1$. Now, following the same reasoning as in Lemmas 9, 10 and 11, it is easy to see that (C.18) holds, that for every $n \geq 0$,

$$\sum_{i=0}^{n-1} p_{i,\mathcal{C}}(q) + p'_{n,\mathcal{C}}(q) = \mathbb{P}_{\mathcal{A}_1}(\mathrm{Cyl}(\pi_{\mathcal{C}}(q_1, e_1))),$$

that $\lim_n p'_{n,\mathcal{C}}(q) = 0$ and thus that

$$\mathbb{P}_{\mathcal{A}_1 \| \mathcal{A}_2}(\mathrm{Cyl}(\pi_{\mathcal{C}^*}(q, \mathcal{A}_2^*, e_1))) = \mathbb{P}_{\mathcal{A}_1}(\mathrm{Cyl}(\pi_{\mathcal{C}}(q_1, e_1))),$$

which concludes the proof. □

We can extend this result to the case where the $n$ first movements in $\mathcal{A}_1$ are determined. For this, we write $\mathrm{Cyl}(\pi_{\mathcal{C}^*}(q, \mathcal{A}_2^*, e_1, \ldots, \mathcal{A}_2^*, e_n)) = \iota_1^{-1}(\mathrm{Cyl}(\pi_{\mathcal{C}}(q_1, e_1, \ldots, e_n)))$ where $\mathcal{C}$ is a Borel set of $\mathbb{R}_+^n$.

**Proposition 5.** *Assuming the previous notations, for every $n \geq 1$, for every $e_1, \ldots, e_n \in E_1$ and for every Borel set $\mathcal{C}$ of $\mathbb{R}_+^n$, we have*

$$\mathbb{P}_{\mathcal{A}_1 \| \mathcal{A}_2}(\mathrm{Cyl}(\pi_{\mathcal{C}^*}(q, \mathcal{A}_2^*, e_1, \ldots, \mathcal{A}_2^*, e_n))) = \mathbb{P}_{\mathcal{A}_1}(\mathrm{Cyl}(\pi_{\mathcal{C}}(q_1, e_1, \ldots, e_n))).$$

We do not give the proof here as it is similar as the proof of Proposition 4. Now, this result can be extended to the elements of the $\sigma$-algebra:

**Proposition 6.** *Assuming the above notations, for every property $\varphi_1$ measurable in $\mathcal{A}_1$ from $q_1$, we have*

$$\mathbb{P}_{\mathcal{A}_1 \| \mathcal{A}_2}(\{q \models \widetilde{\varphi}_1\}) = \mathbb{P}_{\mathcal{A}_1}(\{q_1 \models \varphi_1\}).$$

*Proof.* The proof is immediate from Proposition 5 by noticing that, given a stochastic timed automaton $\mathcal{A}$, the complementary of a cylinder is a countable union of cylinders, that the union of two cylinders $\mathrm{Cyl}(\pi_{\mathcal{C}_1}(q, e_1, \ldots, e_n))$ and $\mathrm{Cyl}(\pi_{\mathcal{C}_2}(q, e_1, \ldots, e_m))$ with $n \leq m$ can be rewritten as

$$\mathrm{Cyl}(\pi_{\mathcal{C}_1}(q, e_1, \ldots, e_n)) \cup \mathrm{Cyl}(\pi_{\mathcal{C}_2 \setminus \mathcal{C}_2^{(n)}}(q, e_1, \ldots, e_m))$$

where $\mathcal{C}_2^{(n)} = \{(t_1, \ldots, t_m) \in \mathcal{C}_2 \mid (t_1, \ldots, t_n) \in \mathcal{C}_1\}$, which is the union of two disjoint cylinders, and by noticing that for every sequence $(A_n)_{n \geq 0} \subseteq \Omega^q_{\mathcal{A}}$ and for every $A \in \Omega^q_{\mathcal{A}}$, we have

$$\iota^{-1}\Big(\bigcup_{n \geq 0} A_n\Big) = \bigcup_{n \geq 0} \iota^{-1}(A_n) \quad \text{and} \quad \iota^{-1}(A^c) = \iota^{-1}(A)^c.$$

□

Similar results as Propositions 4, 5 and 6 hold when we alternate $\mathcal{A}_1$ and $\mathcal{A}_2$. These propositions will lead to (2) but before getting to that, we need an extra notion.

**Definition 5.** *Let $e_1, \ldots, e_n$ (resp. $f_1, \ldots, f_m$) be edges of $\mathcal{A}_1$ (resp. $\mathcal{A}_2$) and let $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) be Borel sets of $\mathbb{R}^n_+$ (resp. $\mathbb{R}^m_+$). We define the shuffle of the cylinders $\mathrm{Cyl}(\pi_{\mathcal{C}_1}(q_1, e_1, \ldots, e_n))$ and $\mathrm{Cyl}(\pi_{\mathcal{C}_2}(q_2, f_1, \ldots, f_m))$ as the following set of runs of $\mathcal{A}_1 \parallel \mathcal{A}_2$:*

$$\{\rho \in \mathrm{Runs}(\mathcal{A}_1 \parallel \mathcal{A}_2, (q_1, q_2)) \mid \iota_1(\rho) \in \mathrm{Cyl}(\pi_{\mathcal{C}_1}(q_1, e_1, \ldots, e_n)) \\ \wedge \iota_2(\rho) \in \mathrm{Cyl}(\pi_{\mathcal{C}_2}(q_2, f_1, \ldots, f_m))\}.$$

*We denote this set by $\mathrm{Cyl}(\pi_{\mathcal{C}_1}(q_1, e_1, \ldots, e_n)) \sqcup\!\sqcup \mathrm{Cyl}(\pi_{\mathcal{C}_2}(q_2, f_1, \ldots, f_m))$.*

*Remark 8.* The shuffle of two cylinders can be rewritten as a union of disjoint cylinders. As a simple example, assuming the same notations of Definition 5, we have that

$$\mathrm{Cyl}(\pi(q_1, e_1, e_2)) \sqcup\!\sqcup \mathrm{Cyl}(\pi(q_2, f_1)) = \\ \mathrm{Cyl}(\pi(q, f_1, \mathcal{A}_2^*, e_1, \mathcal{A}_2^*, e_2)) \cup \mathrm{Cyl}(\pi(q, e_1, f_1, \mathcal{A}_2^*, e_2)) \cup \mathrm{Cyl}(\pi(q, e_1, e_2, \mathcal{A}_1^*, f_1)).$$

Let us also remark that Definition 5 trivially extends to sets of $\sigma$-algebras $\Omega^{\mathcal{A}_1}_{q_1}$ and $\Omega^{\mathcal{A}_2}_{q_2}$. Hence we can notice that, given two properties $\varphi_1$ and $\varphi_2$ measurable in $\mathcal{A}_1$ and $\mathcal{A}_2$,

$$\{q \models \widetilde{\varphi}_1\} \cap \{q \models \widetilde{\varphi}_2\} = \{q_1 \models \varphi_1\} \sqcup\!\sqcup \{q_2 \models \varphi_2\}.$$

We are now able to prove (2) that is

$$\mathbb{P}_{\mathcal{A}_1\|\mathcal{A}_2}(\{q \models \widetilde{\varphi}_1\} \cap \{q \models \widetilde{\varphi}_2\}) = \mathbb{P}_{\mathcal{A}_1}(\{q_1 \models \varphi_1\}) \cdot \mathbb{P}_{\mathcal{A}_2}(\{q_2 \models \varphi_2\}), \qquad (2)$$

where $\varphi_1$ (resp. $\varphi_2$) is a property measurable in $\mathcal{A}_1$ (resp. $\mathcal{A}_2$) from $q_1$ (resp. $q_2$). As in Proposition 6, since for every properties $\varphi_1$ measurable in $\mathcal{A}_1$ from $q_1$ and $\varphi_2$ measurable in $\mathcal{A}_2$ from $q_2$, we have

$$\{q \models \widetilde{\varphi}_1\} \cap \{q \models \widetilde{\varphi}_2\} = \{q_1 \models \varphi_1\} \sqcup\!\sqcup \{q_2 \models \varphi_2\},$$

and $\{q_i \models \varphi_i\} \in \Omega_{q_i}^{\mathcal{A}_i}$ for $i \in \{1, 2\}$, it suffices to prove that for every $n, m \geq 0$, for every $e_1, \dots, e_n$ edges of $\mathcal{A}_1$, for every $f_1, \dots, f_m$ edges of $\mathcal{A}_2$, for every borel sets $\mathcal{C}_1$ of $\mathbb{R}_+^n$ and for every borel sets $\mathcal{C}_2$ of $\mathbb{R}_+^m$,

$$\begin{aligned}&\mathbb{P}_{\mathcal{A}_1\|\mathcal{A}_2}\big(\operatorname{Cyl}(\pi_{\mathcal{C}_1}(q_1, e_1, \dots, e_n)) \sqcup\!\sqcup \operatorname{Cyl}(\pi_{\mathcal{C}_2}(q_2, f_1, \dots, f_m))\big) = \\ &\quad \mathbb{P}_{\mathcal{A}_1}\big((\operatorname{Cyl}(\pi_{\mathcal{C}_1}(q_1, e_1, \dots, e_n))\big) \cdot \mathbb{P}_{\mathcal{A}_2}\big(\operatorname{Cyl}(\pi_{\mathcal{C}_2}(q_2, f_1, \dots, f_m))\big).\end{aligned} \qquad (C.21)$$

We prove it by induction over $(n, m)$. We can first notice that if $n = 0$, then for every $m \geq 0$,

$$\operatorname{Cyl}(\pi(q_1)) \sqcup\!\sqcup \operatorname{Cyl}(\pi_{\mathcal{C}_2}(q_2, f_1, \dots, f_m)) = \operatorname{Cyl}(\pi_{\mathcal{C}_2^*}(q, \mathcal{A}_1^*, f_1, \dots, \mathcal{A}_1^*, f_m))$$

and thus (C.21) holds from Proposition 5. Hence, equality (C.21) holds for each $(0, m)$ with $m \geq 0$. It can similarly be shown that the property holds for each $(n, 0)$ with $n \geq 0$. Now, we fix $n, m \geq 0$ and we assume that (C.21) is satisfied for every $(n', m')$ with $0 \leq n' \leq n$ and $0 \leq m' \leq m$, and we prove that it is still verified for $(n+1, m+1)$. Let $e_1, \dots, e_{n+1}$ and $f_1, \dots, f_{m+1}$ be edges of $\mathcal{A}_1$ and $\mathcal{A}_2$, and let $\mathcal{C}_1$ and $\mathcal{C}_2$ be Borel sets of $\mathbb{R}_+^{n+1}$ and $\mathbb{R}_+^{m+1}$. Then

$$\begin{aligned}&\mathbb{P}_{\mathcal{A}_1\|\mathcal{A}_2}\big(\operatorname{Cyl}(\pi_{\mathcal{C}_1}(q_1, e_1, \dots, e_{n+1})) \sqcup\!\sqcup \operatorname{Cyl}(\pi_{\mathcal{C}_2}(q_2, f_1, \dots, f_{m+1}))\big) = \\ &\quad \int_{t_1 \in I(q, e_1)} p_{q_1+t_1}(e_1) f_{q_1}(t_1)(1 - F_{q_2}(t_1)) \\ &\mathbb{P}_{\mathcal{A}_1\|\mathcal{A}_2}\big(\operatorname{Cyl}(\pi_{\mathcal{C}_1^{t_1}}(q_{t_1}, e_2, \dots, e_{n+1})) \sqcup\!\sqcup \operatorname{Cyl}(\pi_{\mathcal{C}_2^{t_1,+}}(q_2 + t_1, f_1, \dots, f_{m+1}))\big)\,\mathrm{d}t_1 \\ &+ \int_{t_2 \in I(q, f_1)} p_{q_2+t_2}(f_1) f_{q_2}(t_2)(1 - F_{q_1}(t_2)) \\ &\mathbb{P}_{\mathcal{A}_1\|\mathcal{A}_2}\big(\operatorname{Cyl}(\pi_{\mathcal{C}_1^{t_2,+}}(q_1 + t_2, e_1, \dots, e_{n+1})) \\ &\qquad\qquad \sqcup\!\sqcup \operatorname{Cyl}(\pi_{\mathcal{C}_2^{t_2}}(q_{t_2}, f_2, \dots, f_{m+1}))\big)\,\mathrm{d}t_2,\end{aligned} \qquad (C.22)$$

where $\mathcal{C}_1^{t_1} = \{(\tau_2, \dots, \tau_{n+1}) \mid (t_1, \tau_2, \dots, \tau_{n+1}) \in \mathcal{C}_1\}$ which is a Borel set of $\mathbb{R}_+^n$ and $\mathcal{C}_1^{t_2,+} = \{(\tau_1, \dots, \tau_{n+1}) \mid (t_2 + \tau_1, \tau_2, \dots, \tau_{n+1}) \in \mathcal{C}_1\}$ which is a Borel set of

$\mathbb{R}_+^{n+1}$. This is similar with $\mathcal{C}_2$. Now, by induction hypothesis, we obtain that

$$\int_{t_1 \in I(q,e_1)} p_{q_1+t_1}(e_1) f_{q_1}(t_1)(1 - F_{q_2}(t_1))$$
$$\mathbb{P}_{\mathcal{A}_1 \| \mathcal{A}_2}\big(\operatorname{Cyl}(\pi_{\mathcal{C}_1^{t_1}}(q_{t_1}, e_2, \ldots, e_{n+1})) \sqcup \operatorname{Cyl}(\pi_{\mathcal{C}_2^{t_1,+}}(q_2 + t_1, f_1, \ldots, f_{m+1}))\big)\, \mathrm{d}t_1$$
$$= \int_{t_1 \in I(q,e_1)} p_{q_1+t_1}(e_1) f_{q_1}(t_1)(1 - F_{q_2}(t_1)) \mathbb{P}_{\mathcal{A}_1}\big(\operatorname{Cyl}(\pi_{\mathcal{C}_1^{t_1}}(q_{t_1}, e_2, \ldots, e_{n+1}))\big)$$
$$\mathbb{P}_{\mathcal{A}_2}\big(\operatorname{Cyl}(\pi_{\mathcal{C}_2^{t_1,+}}(q_2 + t_1, f_1, \ldots, f_{m+1}))\big)\, \mathrm{d}t_1$$
$$= \int_{t_1 \in I(q,e_1)} p_{q_1+t_1}(e_1) f_{q_1}(t_1)(1 - F_{q_2}(t_1)) \mathbb{P}_{\mathcal{A}_1}\big(\operatorname{Cyl}(\pi_{\mathcal{C}_1^{t_1}}(q_{t_1}, e_2, \ldots, e_{n+1}))\big)$$
$$\int_{u \in I(q_2+t_1, f_1)} p_{q_2+t_1+u}(f_1) f_{q_2+t_1}(u) \mathbb{P}_{\mathcal{A}_2}\big(\operatorname{Cyl}(\pi_{\mathcal{C}_2^{t_1+u}}(q_{t_1+u}, f_2, \ldots, f_{m+1}))\big)\, \mathrm{d}u\, \mathrm{d}t_1.$$

Now, using similar substitution arguments as in the proof of Lemma 8 by letting $t_2 = t_1 + u$, it follows that

$$\int_{t_1 \in I(q,e_1)} p_{q_1+t_1}(e_1) f_{q_1}(t_1)(1 - F_{q_2}(t_1))$$
$$\mathbb{P}_{\mathcal{A}_1 \| \mathcal{A}_2}\big(\operatorname{Cyl}(\pi_{\mathcal{C}_1^{t_1}}(q_{t_1}, e_2, \ldots, e_{n+1})) \sqcup \operatorname{Cyl}(\pi_{\mathcal{C}_2^{t_1,+}}(q_2 + t_1, f_1, \ldots, f_{m+1}))\big)\, \mathrm{d}t_1$$
$$= \int_{t_1 \in I(q,e_1)} p_{q_1+t_1}(e_1) f_{q_1}(t_1) \mathbb{P}_{\mathcal{A}_1}\big(\operatorname{Cyl}(\pi_{\mathcal{C}_1^{t_1}}(q_{t_1}, e_2, \ldots, e_{n+1}))\big)$$
$$\int_{t_2 \in I(q,f_1)} p_{q_2+t_2}(f_1) f_{q_2}(t_2) \mathbb{1}_{[t_1,+\infty[}(t_2)$$
$$\mathbb{P}_{\mathcal{A}_2}\big(\operatorname{Cyl}(\pi_{\mathcal{C}_2^{t_2}}(q_{t_2}, f_2, \ldots, f_{m+1}))\big)\, \mathrm{d}t_2\, \mathrm{d}t_1. \tag{C.23}$$

Now, still by induction hypothesis and using similar arguments as before, we get that

$$\int_{t_2 \in I(q,f_1)} p_{q_2+t_2}(f_1) f_{q_2}(t_2)(1 - F_{q_1}(t_2))$$
$$\mathbb{P}_{\mathcal{A}_1 \| \mathcal{A}_2}\big(\operatorname{Cyl}(\pi_{\mathcal{C}_1^{t_2,+}}(q_1 + t_2, e_1, \ldots, e_{n+1})) \sqcup \operatorname{Cyl}(\pi_{\mathcal{C}_2^{t_2}}(q_{t_2}, f_2, \ldots, f_{m+1}))\big)\, \mathrm{d}t_2$$
$$= \int_{t_1 \in I(q,e_1)} p_{q_1+t_1}(e_1) f_{q_1}(t_1) \mathbb{P}_{\mathcal{A}_1}\big(\operatorname{Cyl}(\pi_{\mathcal{C}_1^{t_1}}(q_{t_1}, e_2, \ldots, e_{n+1}))\big)$$
$$\int_{t_2 \in I(q,f_1)} p_{q_2+t_2}(f_1) f_{q_2}(t_2) \mathbb{1}_{[0,t_1[}(t_2)$$
$$\mathbb{P}_{\mathcal{A}_2}\big(\operatorname{Cyl}(\pi_{\mathcal{C}_2^{t_2}}(q_{t_2}, f_2, \ldots, f_{m+1}))\big)\, \mathrm{d}t_2\, \mathrm{d}t_1. \tag{C.24}$$

Finally, from (C.22), (C.23) and (C.24), we obtain that

$$\begin{aligned}
&\mathbb{P}_{\mathcal{A}_1\|\mathcal{A}_2}\big(\,\mathrm{Cyl}(\pi_{\mathcal{C}_1}(q_1,e_1,\ldots,e_{n+1}))\sqcup\!\sqcup\,\mathrm{Cyl}(\pi_{\mathcal{C}_2}(q_2,f_1,\ldots,f_{m+1}))\big) = \\
&= \int_{t_1\in I(q,e_1)} p_{q_1+t_1}(e_1) f_{q_1}(t_1)\mathbb{P}_{\mathcal{A}_1}\big(\,\mathrm{Cyl}(\pi_{\mathcal{C}_1^{t_1}}(q_{t_1},e_2,\ldots,e_{n+1}))\big)\,\mathrm{d}t_1 \\
&\quad \int_{t_2\in I(q,f_1)} p_{q_2+t_2}(f_1) f_{q_2}(t_2)\mathbb{P}_{\mathcal{A}_2}\big(\,\mathrm{Cyl}(\pi_{\mathcal{C}_2^{t_2}}(q_{t_2},f_2,\ldots,f_{m+1}))\big)\,\mathrm{d}t_2 \\
&= \mathbb{P}_{\mathcal{A}_1}\big((\mathrm{Cyl}(\pi_{\mathcal{C}_1}(q_1,e_1,\ldots,e_{n+1}))\big)\cdot\mathbb{P}_{\mathcal{A}_2}\big(\,\mathrm{Cyl}(\pi_{\mathcal{C}_2}(q_2,f_1,\ldots,f_{m+1}))\big)
\end{aligned}$$

which concludes the proof of (2).

Now, in order to complete the proof of Theorem 2, it remains to show that $\mathcal{A}_1 \parallel \mathcal{A}_2$ is almost-surely non-Zeno. Let $q_0 = ((l_1, l_2), \mathbf{0}_X)$ be an initial state of $\mathcal{A}_1 \parallel \mathcal{A}_2$ and let $q_0^{(1)} = (l_1, \mathbf{0}_{X_1})$ and $q_0^{(2)} = (l_2, \mathbf{0}_{X_2})$. Let us write $\varphi$ for the property in $\mathcal{A}_1 \parallel \mathcal{A}_2$ saying that a run $\rho \in \mathrm{Runs}(\mathcal{A}_1 \parallel \mathcal{A}_2, q_0)$ is Zeno and $\varphi_1$ (resp. $\varphi_2$) for the property saying that a run $\rho_1 \in \mathrm{Runs}(\mathcal{A}_1, q_0^{(1)})$ (resp. $\rho_2 \in \mathrm{Runs}(\mathcal{A}_2, q_0^{(2)})$) is Zeno. It holds that $\varphi$, $\varphi_1$ and $\varphi_2$ are measurable. Assuming previous notations, we have that

$$\{q_0 \models \varphi\} \subseteq \{q_0 \models \widetilde{\varphi}_1\} \cup \{q_0 \models \widetilde{\varphi}_2\}.$$

Indeed, let $\rho \in \mathrm{Runs}(\mathcal{A}_1 \parallel \mathcal{A}_2, q_0)$ be a Zeno run. Then we can write

$$\rho = q_0 \xrightarrow{\tau_1, e_1} q_1 \xrightarrow{\tau_2, e_2} \ldots$$

where $\sum_{k\geq 1} \tau_k < +\infty$. Since $\rho$ is a terminal run, we have that there is $i \in \{1,2\}$ such that $\{k \geq 1 \mid e_k \in E_i\}$ is an infinite set. Then, we have that

$$\iota_i(\rho) = q_0^{(i)} \xrightarrow{t_1, e_{i_1}} \cdot \xrightarrow{t_2, e_{i_2}} \ldots$$

where $\{i_j \mid j \geq 1\} = \{k \geq 1 \mid e_k \in E_i\}$ and $\sum_{l\geq 1} t_l = \sum_{k\geq 1} \tau_k$ since time-transitions are conserved after projection. Hence, $\iota_i(\rho)$ is Zeno and thus $\rho \in \{q_0 \models \widetilde{\varphi}_i\}$. We thus have $\{q_0 \models \varphi\} \subseteq \{q_0 \models \widetilde{\varphi}_1\} \cup \{q_0 \models \widetilde{\varphi}_2\}$ and then,

$$\begin{aligned}
\mathbb{P}_{\mathcal{A}_1\|\mathcal{A}_2}(\{q_0 \models \varphi\}) &\leq \mathbb{P}_{\mathcal{A}_1\|\mathcal{A}_2}\big(\{q_0 \models \widetilde{\varphi}_1\} \cup \{q_0 \models \widetilde{\varphi}_2\}\big) \\
&\leq \mathbb{P}_{\mathcal{A}_1\|\mathcal{A}_2}(\{q_0 \models \widetilde{\varphi}_1\}) + \mathbb{P}_{\mathcal{A}_1\|\mathcal{A}_2}(\{q_0 \models \widetilde{\varphi}_2\}) \\
&= \mathbb{P}_{\mathcal{A}_1}(\{q_0^{(1)} \models \varphi_1\}) + \mathbb{P}_{\mathcal{A}_2}(\{q_0^{(2)} \models \varphi_2\}) \\
&\quad \text{from Proposition 6} \\
&= 0 \quad \text{since } \mathcal{A}_1 \text{ and } \mathcal{A}_2 \text{ are almost-surely non-Zeno.}
\end{aligned}$$

Thus $\mathbb{P}_{\mathcal{A}_1\|\mathcal{A}_2}(\{q_0 \models \varphi\}) = 0$, i.e. $\mathcal{A}_1 \parallel \mathcal{A}_2$ is almost-surely non-Zeno which concludes the proof. □

# D Details for Section 4

## D.1 Details for Subsection 4.1

**Proposition 2.** *Let $\mathcal{A}$ be a STA and let $\mathcal{R}$ be a bisimulation for $\mathcal{A}$. Then for all $q, q' \in Q$, $q\mathcal{R}q'$ if and only if (i)$\mathcal{L}(q) = \mathcal{L}(q')$, (ii) $\mu_q = \mu_{q'}$, and (iii) for every $C \in \mathrm{pcl}(\mathcal{R})$, $P_{q+t}(C) = P_{q'+t}(C)$ almost-surely for every $t \geq 0$.*

*Proof.* In order to prove this characterisation, we first need this very technical lemma. The proof of this lemma is not given here as it has no interest.

**Lemma 12.** *Let $g_1, g_2, h : \mathbb{R}_+ \to \mathbb{R}_+$ be measurable functions such that there is $J \subseteq \mathbb{R}_+$ with $\Lambda(J) > 0$ and $h$ almost-surely non null on $J$. If for every $I \in \mathcal{B}(\mathbb{R}_+)$,*

$$\int_{t\in I} g_1(t)h(t)\,\mathrm{d}t = \int_{t\in I} g_2(t)h(t)\,\mathrm{d}t, \tag{D.1}$$

*then $g_1 = g_2$ almost-surely on $J$.*

The fact that if $q$ and $q'$ satisfy points (i), (ii) and (iii) then $q\mathcal{R}q'$ comes immediately from Definition 4. Indeed point (i) of Definition 4 is the same statement as point (i) of this proposition. Point (ii) of the same definition comes from point (ii) and (iii) of this proposition and from the fact that $\mathbb{P}_{\mathcal{A}}\big(\{q \models \xrightarrow{I} C\}\big) = \int_{t\in I} P_{q+t}(C) f_q(t)\,\mathrm{d}t$ for each Borel sets $I \subseteq \mathbb{R}_+$ and each $C \in \mathrm{cl}(\mathcal{R})$, where $f_q$ is the density function associated with $\mu_q$. Now let us assume that $q\mathcal{R}q'$. Point (i) trivially holds from Definition 4. We also know that for every $I \in \mathcal{B}(\mathbb{R}_+)$ and for every $C \in \mathrm{cl}(\mathcal{R})$,

$$\int_{t\in I} P_{q+t}(C) f_q(t)\,\mathrm{d}t = \int_{t\in I} P_{q'+t}(C) f_{q'}(t)\,\mathrm{d}t.$$

In particular, with $C = Q \in \mathrm{cl}(\mathcal{R})$, we have $P_{q+t}(C) = 1 = P_{q'+t}(C)$ for every $t$ and thus, for every $I \in \mathcal{B}(\mathbb{R}_+)$,

$$\int_{t\in I} f_q(t)\,\mathrm{d}t = \int_{t\in I} f_{q'}(t)\,\mathrm{d}t$$

which leads to the fact that $f_q = f_{q'}$ almost-surely and thus $\mu_q = \mu_{q'}$. It remains to show point (iii). If we fix $C \in \mathrm{cl}(\mathcal{R})$ we can show that the function $P_{q+\bullet}(C)$, assigning the value $P_{q+t}(C)$ to each real positive number $t$, is measurable. Indeed, we have that

$$P_{q+t}(C) = \sum_{l'\in L} \sum_{e\in E_{l'}} p_{q+t}(e) \mathbb{1}_{\mathcal{C}_{l'}(e,\nu)}(t).$$

We already know that for each $e$, $p_{q+\bullet}(e)$ is measurable. If we assume that $q = (l, \nu)$ with $\nu \in \mathbb{R}^n_+$, then for each $e$ and $l'$, the set $\mathcal{C}_{l'}(e,\nu)$ corresponds to the set $\big([Y \leftarrow 0] \circ g_\nu\big)^{-1}(C_l)$ where $g_\nu : \mathbb{R}_+ \to \mathbb{R}^n_+$ is the function that assigns the value $\nu + t$ for each $t \in \mathbb{R}_+$. Now since $[Y \leftarrow 0]$ and $g_\nu$ are two measurable functions and since $C_l$ is a Borel set, we get that $\mathcal{C}_{l'}(e,\nu)$ is a Borel set. Hence $\mathbb{1}_{\mathcal{C}_{l'}(e,\nu)}(t)$ is a measurable function. We can thus conclude that $P_{q+\bullet}(C)$ is also measurable. The same reasoning can be applied to $P_{q'+\bullet}(C)$. Finally, from Lemma 12 applied to $g_1(t) = P_{q+t}(C)$, $g_2(t) = P_{q'+t}(C)$ and $h = f_q = f_{q'}$ almost-surely, we get that point (iii) is satisfied. ◻

### Construction of a bisimulation between two STA

We have defined a bisimulation as a relation between states of a stochastic timed automaton. We would like now to define a bisimulation as a relation between stochastic timed automata with the same set of atomic propositions $AP$. A classical way to achieve this objective (see [7]), is to consider the disjoint union of two stochastic timed automata and to define a bisimulation between these to automata as a bisimulation for the disjoint union of both automata. We thus need to define the disjoint union of two stochastic timed automata. Let $\mathcal{A}_i = \big(L_i, L_0^{(i)}, X_i, E_i, AP, \mathcal{L}_i, (\mu_q^{(i)}, p_q^{(i)})_{q \in L_i \times \mathbb{R}_+^{X_i}}\big)$ for $i \in \{1,2\}$ be two stochastic timed automata with $L_1 \cap L_2 = \emptyset$, $X_1 \cap X_2 = \emptyset$ and $E_1 \cap E_2 = \emptyset$. The *disjoint union* of $\mathcal{A}_1$ and $\mathcal{A}_2$ is the stochastic timed automaton $\mathcal{A}_1 \cup \mathcal{A}_2$ defined by

$$\mathcal{A}_1 \cup \mathcal{A}_2 = (L, L_0, X, E, AP, \mathcal{L}, (\mu_q, p_q)_{q \in L \times \mathbb{R}_+^X}),$$

where $L = L_1 \cup L_2$, $L_0 = L_0^{(1)} \cup L_0^{(2)}$, $X = X_1 \cup X_2$, $E = E_1 \cup E_2$, $\mu_q$ and $p_q$ are such that if $q = (l_1, \nu_1, \nu_2)$ with $l_1 \in L_1$, $\nu_1 \in \mathbb{R}_+^{X_1}$ and $\nu_2 \in \mathbb{R}_+^{X_2}$ then $\mu_q = \mu_{(l_1,\nu_1)}^{(1)}$ and $p_q = p_{(l_1,\nu_1)}^{(1)}$ and similarly, if $q = (l_2, \nu_1, \nu_2)$ with $l_2 \in L_2$ then $\mu_q = \mu_{(l_2,\nu_2)}^{(2)}$ and $p_q = p_{(l_2,\nu_2)}^{(2)}$, and $\mathcal{L}(l) = \mathcal{L}_1(l)$ if $l \in L_1$ and $\mathcal{L}(l) = \mathcal{L}_2(l)$ if $l \in L_2$.

We can now define the notion of bisimulation between two stochastic timed automata with the same set of atomic propositions.

**Definition 6.** *Let $\mathcal{A}_i = \big(L_i, L_0^{(i)}, X_i, E_i, AP, \mathcal{L}_i, (\mu_q^{(i)}, p_q^{(i)})_{q \in L_i \times \mathbb{R}_+^{X_i}}\big)$ for $i \in \{1,2\}$ be two stochastic timed automata with $L_1 \cap L_2 = \emptyset$, $X_1 \cap X_2 = \emptyset$ and $E_1 \cap E_2 = \emptyset$. An equivalence relation over $Q_1 \cup Q_2$ is a bisimulation between $\mathcal{A}_1$ and $\mathcal{A}_2$ if it is a bisimulation for $\mathcal{A}_1 \cup \mathcal{A}_2$. We say that $\mathcal{A}_1$ and $\mathcal{A}_2$ are bisimilar, and we write $\mathcal{A}_1 \sim \mathcal{A}_2$, if for every $l_0^{(1)} \in L_0^{(1)}$, there is $l_0^{(2)} \in L_0^{(2)}$ such that $q_0^{(1)} \sim q_0^{(2)}$ where $q_0^{(i)} = (l_0^{(i)}, \mathbf{0}_X)$, and vice versa.*

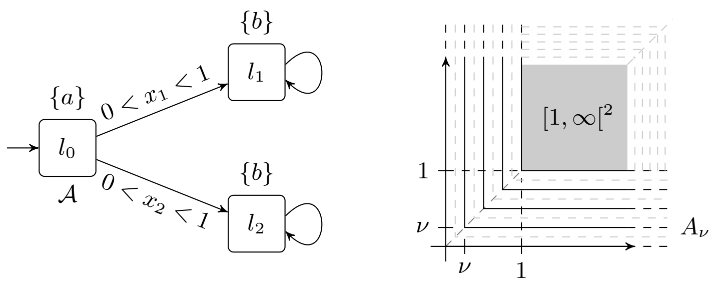
*Remark 9.* Let us note that given two stochastic timed automata $\mathcal{A}_1$ and $\mathcal{A}_2$ and given a state $(l_1, \nu_1, \nu_2)$ of $\mathcal{A}_1 \cup \mathcal{A}_2$ with $l_1 \in L_1$, $\nu_1 \in \mathbb{R}_+^{X_1}$ and $\nu_2 \in \mathbb{R}_+^{X_2}$, it holds that $\forall \nu_2' \in \mathbb{R}_+^{X_2},\ (l_1, \nu_1, \nu_2') \sim (l_1, \nu_1, \nu_2)$. This comes from the fact that from state $(l_1, \nu_1, \nu_2)$ in the union, with $l_1 \in L_1$, only edges of $\mathcal{A}_1$ are enabled. Hence, the state $(l_1, \nu_1, \nu_2)$ will behave in $\mathcal{A}_1 \cup \mathcal{A}_2$ exactly as state $(l_1, \nu_1)$ in $\mathcal{A}_1$. Thus, the value of $\nu_2$ has no impact on the behaviour of $\mathcal{A}_1 \cup \mathcal{A}_2$ from $(l_1, \nu_1, \nu_2)$. The same reasoning applies to states $(l_2, \nu_1, \nu_2)$ with $l_2 \in L_2$. A state of the form $(l_i, \nu_1, \nu_2)$ with $l_i \in L_i$ will then be abusively identified as the state $(l_i, \nu_i)$.

One can also observe that our notion of bisimulation does not match with the notion of *strong timed* bisimulation, that is two states $q$ and $q'$ are strong timed bisimilar whenever the following statement holds: if there is $t \geq 0$ and an edge $e$ such that $q \xrightarrow{t,e} q_1$, then there is an edge $e'$ such that $q' \xrightarrow{t,e'} q_1'$ with $q_1$ and $q_1'$ strong timed bisimilar, and vice versa. Indeed, let us consider two simple stochastic timed automata $\mathcal{A}_1$ and $\mathcal{A}_2$, such that for each $i = 1, 2$ there is a single clock $x_i$, there is only one edge $e_i$ relying $l_i$ to $l'_i$, with the only difference that the guard of edge $e_1$ is $x_1 < 1$ and the guard of edge $e_2$ is $0 < x_2 < 1$, and such that $l_1$, $l_2$ have the same labels and $l'_1$, $l'_2$ have the same labels. Then, it is easy to show that $(l_1, 0)$ and $(l_2, 0)$ are bisimilar. However, there are not strong timed bisimilar as in $\mathcal{A}_1$ you can perform $(l_1, 0) \xrightarrow{0,e_1} (l'_1, 0)$ while you cannot perform $(l_2, 0) \xrightarrow{0,e_2} (l'_2, 0)$ in $\mathcal{A}_2$

**Details for Examples 5 and 6**

Let us consider the simple STA $\mathcal{A}$ with two clocks on Figure 5 (repeated below). We assume that each state of the form $(l_1, \nu)$ or $(l_2, \nu')$ with $\nu$, $\nu' \in \mathbb{R}_+^2$ is equipped with the same exponential distribution over delays, say $\mathrm{Exp}(\lambda)$. Now, from a state of the form $q = (l_0, (\nu_1, \nu_2))$ with $\nu_1 < 1$ or $\nu_2 < 1$, $I(q) = [0, 1 - \min(\nu_1, \nu_2)[$ and so we can equip $q$ with a uniform distribution on the interval $I(q)$ for the delays.

We now compute the equivalence classes for $\sim$. It can easily be established that the set of states $\{l_1, l_2\} \times \mathbb{R}_+^2$ is an equivalence class for $\sim$. In order to find



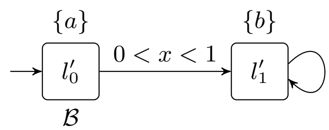

A simple example for bisimulation.

the equivalence classes associated with $l_0$, we use Proposition 2. It is obvious that the set $\{l_0\} \times [1, \infty[^2$ forms an equivalence class for $\sim$ as there are the only states from which no edges are enabled. We can then show that for each $\nu \in [0, 1[$,

$$A_\nu = \{l_0\} \times \big(\{(\nu_1, \nu) \mid \nu_1 \geq \nu\} \cup \{(\nu, \nu_2) \mid \nu_2 \geq \nu\}\big)$$

is an equivalence class. First let us prove that given $q = (l_0, (\nu_1, \nu_2))$ and $q' = (l_0, (\nu'_1, \nu'_2)) \in A_\nu$, both states satisfy point (i), (ii) and (iii) of the characterisation. Point (i) is obvious. In order to get point (ii), it suffices to observe that $\min(\nu_1, \nu_2) = \min(\nu'_1, \nu'_2)$. Finally, for point (iii), From location $l_0$, the only set that is reachable in one step is $C = \{l_1, l_2\} \times \mathbb{R}_+^2$, and we get that for each $t \geq 0$, $P_{q+t}(C) = \mathbb{1}_{I(q)}(t)$ and $P_{q'+t}(C) = \mathbb{1}_{I(q')}(t)$. But since $\min(\nu_1, \nu_2) = \min(\nu'_1, \nu'_2)$, we have $I(q) = I(q')$ and thus $P_{q+t}(C) = P_{q'+t}(C)$. Finally $A_\nu$ is an equivalence class since, if $q = (l_0, (\nu_1, \nu_2)) \in A_\nu$ and $q' = (l_0, (\nu'_1, \nu'_2)) \in A_{\nu'}$ with $\nu \neq \nu'$, then $\min(\nu_1, \nu_2) \neq \min(\nu'_1, \nu'_2)$ and thus point (ii) does not hold.

Now, let us consider the single clock STA $\mathcal{B}$ of Figure 6 (repeated below). Assuming that we have the same probability distributions as STA $\mathcal{A}$ (i.e. uniform distribution for $l'_0$ and Exp($\lambda$)-distribution for $l'_1$), it can be easily established that $\mathcal{B} \sim \mathcal{A}$. It is obvious that each state of $\{l_1, l_2\} \times \mathbb{R}_+^2$ is bisimilar to each state of $\{l'_1\} \times \mathbb{R}_+$ and that each state of $\{l_0\} \times [1, \infty[^2$ is bisimilar to each state of $\{l'_O\} \times [1, \infty[$. Finally, by a similar argument as before, it can be easily established that for each $\nu \in [0, 1[$, $(l'_0, \nu)$ is bisimilar to each state of $A_\nu$.




$\mathcal{B}$ is bisimilar to $\mathcal{A}$.

### D.2 Details for Subsection 4.2

With the aim of proving Lemma 2, we first show the following property describing the nice behaviour of bisimulation w.r.t. delays.

**Lemma 13.** *Let $\mathcal{A}$ be a stochastic timed automaton in* CSTA$^*$ *and let $\mathcal{R}$ be a bisimulation for $\mathcal{A}$. Let $q$ and $q'$ be two states of $\mathcal{A}$. If $q\mathcal{R}q'$ then $(q+t)\mathcal{R}(q'+t)$ for every $t \geq 0$.*

*Proof.* Let $q$ and $q'$ be states of $\mathcal{A}$ such that $q\mathcal{R}q'$. We have to prove that for every $t \geq 0$, $(q+t)\mathcal{R}(q'+t)$. Let $t \geq 0$, we have to show that $q+t$ and $q'+t$ satisfy points (i), (ii) and (iii) of Proposition 2, i.e. that (i) $\mathcal{L}(q+t) = \mathcal{L}(q'+t)$, that (ii) $\mu_{q+t} = \mu_{q'+t}$ and that (iii) for every $C \in \mathrm{cl}(\mathcal{R})$ and for almost every $t' \geq 0$, $P_{(q+t)+t'}(C) = P_{(q'+t)+t'}$. Point (i) is trivial as it only depends on the locations. Point (iii) comes from the fact that for every $C \in \mathrm{cl}(\mathcal{R})$, $P_{q+t''}(C) = P_{q'+t''}(C)$ for almost every $t'' \geq 0$, as $q\mathcal{R}q'$. We thus get that for every $C \in \mathrm{cl}(\mathcal{R})$, $P_{(q+t)+t'}(C) = P_{(q'+t)+t'}(C)$ almost-surely for every $t' \geq 0$. It remains to establish point (ii), i.e. that $\mu_{q+t} = \mu_{q'+t}$. Since $q\mathcal{R}q'$, we already know that $\mu_q = \mu_{q'}$. Then, since from the definition of stochastic timed automaton we have that $\mu_q$ is equivalent to the restriction of the Lebesgue measure over $I(q)$ (denoted $\Lambda_{I(q)}$) and $\mu_{q'}$ is equivalent to $\Lambda_{I(q')}$, we get that

$$\Lambda\big(\{t \in \mathbb{R}_+ \mid (t \in I(q) \wedge t \notin I(q')) \vee (t \notin I(q) \wedge t \in I(q'))\}\big) = 0.$$

From this last statement and from Lemma 3, it can also be shown that $\sup(I(q)) = \sup(I(q'))$. Finally, let us note that since $\mathcal{A} \in$ CSTA$^*$ we have that for every $t' \geq 0$,

$$f_q(t+t') = (1 - F_q(t))f_{q+t}(t') \quad \text{and} \quad f_{q'}(t+t') = (1 - F_{q'}(t))f_{q'+t}(t') \quad \text{(D.2)}$$

where $f_q$, $f_{q'}$, $f_{q+t}$ and $f_{q'+t}$ are the density functions of respectively $\mu_q$, $\mu_{q'}$, $\mu_{q+t}$ and $\mu_{q'+t}$. In order to prove that $\mu_{q+t} = \mu_{q'+t}$, we show that $I(q+t) = \emptyset$ iff $I(q'+t) = \emptyset$. W.l.o.g., let us assume that $I(q+t) \neq \emptyset$ and $I(q'+t) = \emptyset$. Then from Lemma 3, there is $t' \in I(q+t)$ and $\varepsilon > 0$ such that $[t', t'+\varepsilon[ \subseteq I(q+t)$. It thus holds that $[t+t', t+t'+\varepsilon[ \subseteq I(q)$. Then, $\sup(I(q)) \geq t+t'+\varepsilon$ while $\sup(I(q') \leq t$ as $I(q'+t) = \emptyset$. This leads to a contradiction since $\sup(I(q)) = \sup(I(q'))$, allowing us to conclude that $I(q+t) = \emptyset$ iff $I(q'+t) = \emptyset$. In this case, $\mu_{q+t}$ and $\mu_{q'+t}$ are not defined. This suffices to establish that $\Lambda(I(q+t)) > 0$ iff $\Lambda(I(q'+t)) > 0$ and thus, from (D.2) and from the fact that $\mu_q = \mu_{q'}$, we have that $\mu_{q+t} = \mu_{q'+t}$. □

**Lemma 2.** *Let $\mathcal{A}, \mathcal{B} \in$ CSTA$^*$ with sets of states resp. $Q_A$ and $Q_B$. If $\mathcal{R}$ is a bisimulation for $\mathcal{A}$ then the equivalence relation $\mathcal{R}'$ over $Q_A \times Q_B$ defined by $\mathcal{R}' = \{((q_1, q), (q_2, q)) \mid q_1 \mathcal{R} q_2$ and $q \in Q_B\}$, is a bisimulation for $\mathcal{A} \parallel \mathcal{B}$.*

This result seems very intuitive, however the proof is quite technical. The tricky part comes when verifying if for each states $q$ and $q'$ of $\mathcal{A} \parallel \mathcal{B}$ with $q\mathcal{R}'q'$, it holds that for each polyhedral set $C \in \mathrm{pcl}(\mathcal{R}')$ and for almost every $t \geq 0$, $P_{q+t}(C) = P_{q'+t}(C)$. The key part to prove this point, is to decompose and to project $C$ in a way that we can use the fact that $\mathcal{R}$ is a bisimulation for $\mathcal{A}$. This is where we need polyhedral sets for $C$ and we need the measurability of the discrete probability distributions over edges.

We write $\mathcal{A} \parallel \mathcal{B} = \big(L_A \times L_B, (l_0^A, l_0^B), X_A \cup X_B, E_A \cup E_B, (\mu_q, p_q)_{q \in Q_A \times Q_B}, AP, \mathcal{L}\big)$. Given a state $(q_i, q)$ of $\mathcal{A} \parallel \mathcal{B}$, we write $f_{q_i,A}$ for the density function associated with $\mu^A_{q_i}$, $f_{q,B}$ for the density function associated with $\mu^B_q$ and $f_{q_i,min}$ for the density function of $\mu_{(q_i,q)}$ in the product.

*Proof.* Let $q_1 = (l_1, \nu_1)$ and $q_2 = (l_2, \nu_2)$ be states of $\mathcal{A}$ and let $q = (l, \nu)$ be a state of $\mathcal{B}$ such that $(q_1, q)\mathcal{R}'(q_2, q)$, i.e. such that $q_1 \mathcal{R} q_2$. Then, from Proposition 2 it holds that $\mathcal{L}_A(q_1) = \mathcal{L}_A(q_2)$, that $f_{q_1,A} = f_{q_2,A}$ almost-surely and that for each $C \in \mathrm{pcl}(\mathcal{R})$, $P^A_{q_1+\bullet}(C) = P^A_{q_2+\bullet}(C)$ almost-surely. From the definition of the composition we can immediately deduce that $\mathcal{L}((q_1, q)) = \mathcal{L}((q_2, q))$ and that $f_{q_1,min} = f_{q_2,min}$ almost-surely. It remains to show that for each $C \in \mathrm{pcl}(\mathcal{R}')$, $P_{(q_1,q)+\bullet}(C) = P_{(q_2,q)+\bullet}(C)$ almost-surely. Let $C \in \mathrm{pcl}(\mathcal{R}')$, we can write

$$C = \bigcup_{(l_A, l_B) \in L_A \times L_B} \{(l_A, l_B)\} \times C_{(l_A, l_B)}$$

where for each $(l_A, l_B) \in L_A \times L_B$, $C_{(l_A, l_B)}$ is a polyhedral set. Let us first compute the value $P_{(q_1,q)+t}(C)$. We have that

$$\begin{aligned} P_{(q_1,q)+t}(C) = {} & \frac{f_{q_1,A}(t)(1 - F_{q,B}(t))}{f_{q_1,min}(t)} \sum_{l_A \in L_A} \sum_{e_A \in E_{l_A}} p^A_{q_1+t}(e_A) \mathbb{1}_{\mathcal{C}_{(l_A,l)}(e_A,\nu_1,\nu)}(t) \\ & + \frac{f_{q,B}(t)(1 - F_{q_1,A}(t))}{f_{q_1,min}(t)} \sum_{l_B \in L_B} \sum_{e_B \in E_{l_B}} p^B_{q+t}(e_B) \mathbb{1}_{\mathcal{C}_{(l_1,l_B)}(e_B,\nu_1,\nu)}(t), \end{aligned}$$

where, if $e_A = (l_1, g_A, Y_A, l_A)$ and $e_B = (l, g_B, Y_B, l_B)$ then

– $\mathcal{C}_{(l_A,l)}(e_A,\nu_1,\nu) = \{t \in \mathbb{R}_+ \mid \big([Y_A \leftarrow 0](\nu_1+t), \nu+t\big) \in C_{(l_A,l)}\}$, and
– $\mathcal{C}_{(l_1,l_B)}(e_B,\nu_1,\nu) = \{t \in \mathbb{R}_+ \mid \big(\nu_1+t, [Y_B \leftarrow 0](\nu+t)\big) \in C_{(l_1,l_B)}\}$.

Computing the value $P_{(q_2,q)+t}(C)$ is similar. In order to show that for almost-surely every $t \geq 0$, $P_{(q_1,q)+t}(C) = P_{(q_2,q)+t}(C)$, it suffices to prove that for almost-surely every $t \geq 0$.

$$\begin{aligned} w^A_{(q_1,q)}(t) &\sum_{l_A \in L_A} \sum_{e_A \in E_{l_A}} p^A_{q_1+t}(e_A) \mathbb{1}_{\mathcal{C}_{(l_A,l)}(e_A,\nu_1,\nu)}(t) \\ &= w^A_{(q_2,q)}(t) \sum_{l_A \in L_A} \sum_{e_A \in E_{l_A}} p^A_{q_2+t}(e_A) \mathbb{1}_{\mathcal{C}_{(l_A,l)}(e_A,\nu_2,\nu)}(t) \end{aligned} \tag{D.3}$$

and that

$$\begin{aligned} w^B_{(q_1,q)}(t) &\sum_{l_B \in L_B} \sum_{e_B \in E_{l_B}} p^B_{q+t}(e_B) \mathbb{1}_{\mathcal{C}_{(l_1,l_B)}(e_B,\nu_1,\nu)}(t) \\ &= w^B_{(q_2,q)}(t) \sum_{l_B \in L_B} \sum_{e_B \in E_{l_B}} p^B_{q+t}(e_B) \mathbb{1}_{\mathcal{C}_{(l_2,l_B)}(e_B,\nu_2,\nu)}(t). \end{aligned} \tag{D.4}$$

First of all, let us observe that we already have that $w^A_{(q_1,q)} = w^A_{(q_2,q)}$ almost-surely and that $w^B_{(q_1,q)} = w^B_{(q_2,q)}$ since by hypothesis, $f_{q_1,A} = f_{q_2,A}$ almost-surely. Now, we can show that for each $l_B \in L_B$ and for each $e_B = (l, g_B, Y_B, l_B) \in E_{l_B}$, $\mathcal{C}_{(l_1,l_B)}(e_B,\nu_1,\nu) = \mathcal{C}_{(l_2,l_B)}(e_B,\nu_2,\nu)$ almost-surely. Indeed, for almost every $t \geq 0$, $(q_1+t)\mathcal{R}(q_2+t)$ from Lemma 13. Now if $t \in \mathcal{C}_{(l_1,l_B)}(e_B,\nu_1,\nu)$ and $t$ is such that $(q_1+t)\mathcal{R}(q_2+t)$, then $\big(q_1+t, [Y_B \leftarrow 0](q+t)\big) \in C$. Since $\big(q_1+t, [Y_B \leftarrow 0](q+t)\big)\mathcal{R}'\big(q_2+t, [Y_B \leftarrow 0](q+t)\big)$ and $C \in \mathrm{pcl}(\mathcal{R}')$, we get that $\big(q_2+t, [Y_B \leftarrow 0](q+t)\big) \in C$, i.e. $t \in \mathcal{C}_{(l_2,l_B)}(e_B,\nu_2,\nu)$. By a similar argument, we get the almost-sure equality between the two sets. This proves equality (D.4).
In order to show equality (D.3) we first introduce some notation. We write $C_q^{(t)}$ for the following set of states in $\mathcal{A}$: $\{q_A \in Q_A \mid (q_A, q+t) \in C\}$. Equality (D.3) can thus be rewritten as

$$P^A_{q_1+t}(C_q^{(t)}) = P^A_{q_2+t}(C_q^{(t)}). \tag{D.5}$$

We will show that this equality holds almost-surely on each interval $I$ of the form $[a,b]$ with $a$, $b \in \mathbb{R}_+$. Let $I = [a,b]$ be such an interval. For each $n \in \mathbb{N}$, we write

$$I = \bigcup_{k=0}^{2^n-1} I_k^{(n)}$$

where for each $k$, $I_k^{(n)} = \big[a + \frac{k(b-a)}{2^n}, a + \frac{(k+1)(b-a)}{2^n}\big]$. For each $n \in \mathbb{N}$ and each $0 \leq k \leq 2^n - 1$ we write

$$C_q^{(k,n)} = \{q_A \in Q_A \mid \exists t \in I_k^{(n)},\ (q_A, q+t) \in C\}.$$

We can prove that $C_q^{(k,n)} \in \mathrm{pcl}(\mathcal{R})$. Indeed if $q_A \in C_q^{(k,n)}$, then there is $t \in I_k^{(n)}$ such that $(q_A, q+t) \in C$. Let $q'_A$ be such that $q_A \mathcal{R} q'_A$. Then, from definition of $\mathcal{R}'$, we have $(q_A, q+t)\mathcal{R}'(q'_A, q+t)$ and thus $(q'_A, q+t) \in C$ since $C \in \mathrm{pcl}(\mathcal{R}')$. Hence, $q'_A \in C_q^{(k,n)}$. It remains to show that for each $l_A \in L_A$, there is a polyhedral set $C_{l_A}$ such that

$$C_q^{(k,n)} = \bigcup_{l_A \in L_A} \{l_A\} \times C_{l_A}.$$

Such sets exist and are defined as follows. For each $l_A \in L_A$, we define $C_{l_A}$ as follows

$$C_{l_A} = \mathrm{Proj}_{X_A}\left(C_{(l_A,l)} \cap \left(\mathbb{R}_+^{X_A} \times \{\nu + t \mid t \in I_k^{(n)}\}\right)\right)$$

(where $\mathrm{Proj}_{X_A}$ denotes the projection of $\mathbb{R}_+^{X_A \cup X_B}$ over $\mathbb{R}_+^{X_A}$) which is a polyhedral set. Indeed, we have that for each $n$ and $k$, $C_{(l_A,l)}$ and $\mathbb{R}_+^{X_A} \times \{\nu+t \mid t \in I_k^{(n)}\}$ are two polyhedral sets. Then the intersection is still a polyhedral set. And since the projection of a polyhedral set is a polyhedral set, we get that $C_{l_A}$ is a polyhedral set. Which proves that $C_q^{(k,n)} \in \mathrm{pcl}(\mathcal{R})$. It follows that for each $n \geq 0$, for each $0 \leq k \leq 2^n - 1$,

$$P^A_{q_1+t}(C_q^{(k,n)}) = P^A_{q_2+t}(C_q^{(k,n)}) \tag{D.6}$$

for almost every $t \geq 0$. We will now establish that for almost each $t \in I$,

$$\sum_{k=0}^{2^n-1} P^A_{q_1+t}(C_q^{(k,n)}) \mathbb{1}_{I_k^{(n)}}(t) \underset{n\to\infty}{\longrightarrow} P^A_{q_1+t}\big(C_q^{(t)}\big). \tag{D.7}$$

We fix $t \in I$ such that $t$ is not in $\{a + \frac{k(b-a)}{2^n} \mid n \in \mathbb{N} \wedge k \in \{0, \ldots, 2^n\}\}$. It can be shown that $\Lambda(\{a + \frac{k(b-a)}{2^n} \mid n \in \mathbb{N} \wedge k \in \{0, \ldots, 2^n\}\}) = 0$. Then, by construction, for each $n \geq 0$ there is a unique $k_{t,n} \in \{0, \ldots, 2^n - 1\}$ such that $t \in I_{k_{t,n}}^{(n)}$. We thus have to show that

$$P^A_{q_1+t}(C_q^{(k_{t,n},n)}) \underset{n\to\infty}{\longrightarrow} P^A_{q_1+t}\big(C_q^{(t)}\big). \tag{D.8}$$

By construction, we have that for each $n \geq 0$, $I_{k_{t,n+1}}^{(n+1)} \subseteq I_{k_{t,n}}^{(n)}$ and thus $C_q^{(k_{t,n+1},n+1)} \subseteq C_q^{(k_{t,n},n)}$. It follows that

$$C_q^{(k_{t,n},n)} \underset{n\to\infty}{\searrow} \bigcap_{n\geq 0} C_q^{(k_{t,n},n)} = C_q^{(t)}.$$

Since $P^A_{q_1+t}$ is a probability measure over $Q_A$ we conclude, from a classical result, that equality (D.8) holds which proves (D.7). Similarly, we can prove that

$$\sum_{k=0}^{2^n-1} P^A_{q_2+t}(C_q^{(k,n)}) \mathbb{1}_{I_k^{(n)}}(t) \underset{n\to\infty}{\longrightarrow} P^A_{q_2+t}\big(C_q^{(t)}\big). \tag{D.9}$$

for almost every $t \in I$. Now, from (D.6) we get that

$$\sum_{k=0}^{2^n-1} P^A_{q_1+t}(C_q^{(k,n)}) \mathbb{1}_{I_k^{(n)}}(t) = \sum_{k=0}^{2^n-1} P^A_{q_2+t}(C_q^{(k,n)}) \mathbb{1}_{I_k^{(n)}}(t),$$

for almost every $t \in I$, and thus (D.7) and (D.9) allow us to state that equality (D.5) holds almost-surely for each $t \in I$. Since it holds for each interval $I \subseteq \mathbb{R}_+$ of the form $[a, b]$, we get that (D.5) holds almost-surely for each $t \geq 0$ which terminates the proof of the lemma. □

**Theorem 3.** *Bisimulation is a congruence w.r.t. parallel composition. That is: if $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{B}$ are three STA in* CSTA$^*$, *if $\mathcal{A}_1 \sim \mathcal{A}_2$ then $\mathcal{A}_1 \parallel \mathcal{B} \sim \mathcal{A}_2 \parallel \mathcal{B}$.*

*Proof.* There is a bisimulation $\mathcal{R}$ for $\mathcal{A}_1 \cup \mathcal{A}_2$ for each initial state $q_0^{(1)}$ of $\mathcal{A}_1$, there is an initial state $q_0^{(2)}$ of $\mathcal{A}_2$ such that $q_0^{(1)} \mathcal{R} q_0^{(2)}$ and vice versa. We define

$$\mathcal{R}' = \{((q_1, q), (q_2, q)) \mid (q_1 \mathcal{R} q_2) \wedge q \in Q\}$$

and we show that it is a bisimulation between $\mathcal{A}_1 \parallel \mathcal{B}$ and $\mathcal{A}_2 \parallel \mathcal{B}$. Let $((q_0^{(1)}, q_0)$ be an initial state of $\mathcal{A}_1 \parallel \mathcal{B}$. By hypothesis, there is an initial state $q_0^{(2)}$ of $\mathcal{A}_2$ such that $q_0^{(1)} \mathcal{R} q_0^{(2)}$. Let us show that $(q_0^{(1)}, q_0) \mathcal{R}' (q_0^{(2)}, q_0)$. This last statement trivially holds since $q_0^{(1)} \mathcal{R} q_0^{(2)}$. From Lemma 2 we get that $\mathcal{R}'$ is a bisimulation for $(\mathcal{A}_1 \cup \mathcal{A}_2) \parallel \mathcal{B}$. The remark on page 45 allows us then to conclude that $\mathcal{R}'$ is a bisimulation for $(\mathcal{A}_1 \parallel \mathcal{B}) \cup (\mathcal{A}_2 \parallel \mathcal{B})$. □