k-fault tolerant Nash equilibria

Patricia Bouyer[1], Thomas Brihaye[2], Quentin Hautem[2], Nicolas Markey[1]

[1]ENS Cachan, France & [2]University of Mons, Belgium

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Suspect Game

Concurrent game n players $\rightsquigarrow$ zero-sum turn-based game (Suspect game)
Nash equilibrium $\longleftrightarrow$ Winning strategy*
* Romain Brenguier, Nash equilibria in concurrent games - Application to timed games. Doctoral thesis, ENS Cachan, 2012.
Adaptation : Suspect Game $\rightsquigarrow$ Deviator Game

- Maximum use of its capacities;
- deals with several solution concepts;
- Deviator game for NE & Deviator game for k-fault tolerant NE.

Game theory

Need for verifying that computer systems are correct
$\rightsquigarrow$ Model-checking $\rightsquigarrow$ Game theory

Some classical hypotheses in game theory:

- players are *selfish*,
- players are *rational*.

Game theory

- The hypothesis of rationality can be discussed,
- computer systems are not free from failure,
- incompatibility of the rationality,
- introduction of a new solution concept.

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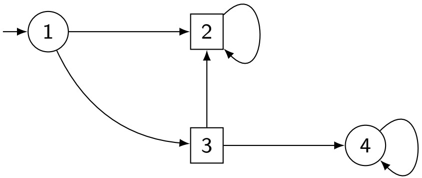
Qualitative turn-based game

$\mathcal{G} = (Agt, V, (V_A)_{A \in Agt}, E, (\preccurlyeq_A)_{A \in Agt})$ where

- Agt is the set of players,
- (V, E) is a finite graph where V is the set of nodes and $E \subseteq V \times V$ is the set of edges,
- $(V_A)_{A \in Agt}$ is a partition of V such that V_A is the set of nodes controlled by player A,
- for all $A \in Agt$, $\preccurlyeq_A$ is a preorder over the terminal histories, called the preference relation of player A.

Example of an arena



Player A ○
Player B □

Nash equilibrium

$Hist_A$ = the set of histories h that ends in a node of V_A.

A pure strategy of player A is a function $\sigma_A : Hist_A \rightarrow V$.

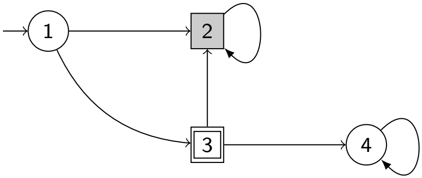
Definition

Let $\mathcal{G} = (Agt, V, (V_A)_{A \in Agt}, E, (\preccurlyeq_A)_{A \in Agt})$ be a turn-based game, v be the initial node, and σ_{Agt} be a strategy profile. Then, σ_{Agt} is a *Nash equilibrium* in $\mathcal{G}$ from v if

$$\forall A \in Agt, \forall \sigma'_A, Out(v, \sigma'_A, \sigma_{-A}) \preccurlyeq_A Out(v, \sigma_{Agt}).$$

$\longrightarrow$ σ_{Agt} is a Nash equilibrium if no player has a profitable deviation.

Examples (1/3)

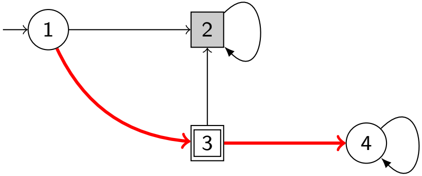


Player A ○, $Reach_A = \{2\}$
Player B □, $Reach_B = \{3\}$

Examples (2/3)

$\sigma_A(1) = 3$
$\sigma_B(13) = 4$

Player A ○, $Reach_A = \{2\}$
Player B □, $Reach_B = \{3\}$

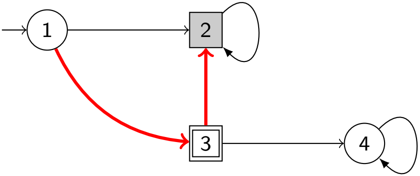


$Out(1, \sigma_A, \sigma_B) = 134^\omega \notin \Omega_A$.
(σ_A, σ_B) is not a Nash equilibrium.

Examples (3/3)

$\sigma_A(1) = 3$
$\sigma'_B(13) = 2$

Player A ○, $Reach_A = \{2\}$
Player B □, $Reach_B = \{3\}$



$Out(1, \sigma_A, \sigma'_B) = 132^\omega$
(σ_A, σ'_B) Nash equilibrium

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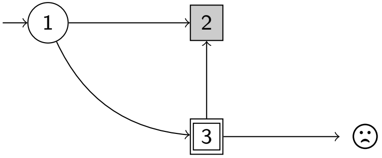
k-fault tolerant

Definition

Let $\mathcal{G} = (Agt, V, (V_A)_{A \in Agt}, E, (\preccurlyeq_A)_{A \in Agt})$ be a turn-based game and $\mathcal{B} \subseteq Plays$ be the subset of plays which are bad for all the players.
Given an integer $k \geq 1$ and v an initial node, we say that a strategy profile σ_{Agt} is a *k-fault tolerant Nash equilibrium* if σ_{Agt} is a *Nash Equilibrium* that satisfies

$$\forall C \subseteq Agt,\ |C| \leq k,\ \forall \sigma'_C \in \text{Strat}_C, \text{Out}(v, \sigma_{-C}, \sigma'_C) \notin \mathcal{B}.$$

Examples (1/3)



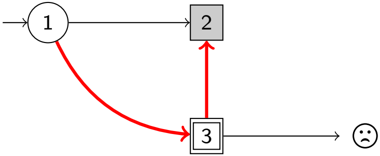
Player A ○, $Reach_A = \{2\}$
Player B □, $Reach_B = \{3\}$
$\beta =$ ☹

Examples (2/3)

$\sigma_A(1) = 3$
$\sigma'_B(13) = 2$

Player A ○, $Reach_A = \{2\}$
Player B □, $Reach_B = \{3\}$

$\beta =$ ☹



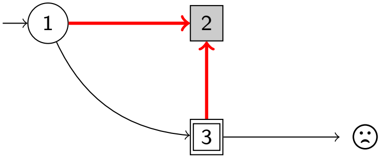
(σ_A, σ'_B) not a 1-fault tolerant Nash equilibrium

Examples (3/3)

$\tau_A(1) = 2$
$\tau_B(13) = 2$

Player A ○, $Reach_A = \{2\}$
Player B □, $Reach_B = \{3\}$

$\beta =$ ☹



(τ_A, τ_B) is a 1-fault tolerant Nash equilibrium.

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Adaptation of the Suspect game

Question : Is there a Nash equilibrium whose outcome is ρ ?
Input : Concurrent game $\mathcal{G}$ and ρ
$\rightsquigarrow$ Suspect Game

Theorem (R. Brenguier, 2012*)

There exists a Nash equilibrium in $\mathcal{G}$ whose outcome is ρ if and only if Eve has a winning strategy in $\mathcal{H}(\mathcal{G}, \rho)$ whose outcome η when Adam obeys is such that $\pi_1(\eta) = \rho$.

* Romain Brenguier, Nash equilibria in concurrent games - Application to timed games. Doctoral thesis, ENS Cachan, 2012.

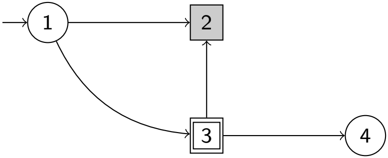
Deviator

Definition

Given a turn-based game, an edge (s, v_1) and a successor v_2 of s, we define

$$Dev((s, v_1), v_2) = \begin{cases} \emptyset & \text{if } v_1 = v_2 \\ \{A\} & \text{if } v_1 \neq v_2 \text{ and } s \in V_A \end{cases}$$

Example



$Dev((1,2),3) = \{A\}, Dev((1,2),2) = \emptyset$ and $Dev((3,2),4) = \{B\}$.

Deviator Game

Input : Turn-based game $\mathcal{G}$ + a play ρ in $\mathcal{G}$
The Deviator game $\mathcal{H}(\mathcal{G}, \rho)$ is a zero-sum turn-based game, played by Adam and Eve.
The game is played in the following way:

1. from a configuration (s, D), Eve chooses a successor v_1 of s;
2. the next state is (s, D, v_1);
3. then Adam chooses a successor v_2 of s;
4. Eve's new state is $(v_2, D \cup Dev((s, v_1), v_2))$.

Remark

When the state v_2 chosen by Adam in step 3 is such that $v_2 = v_1$, we say that Adam **obeys** Eve.

Construction of the deviator game

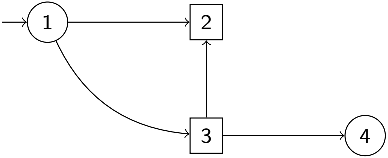
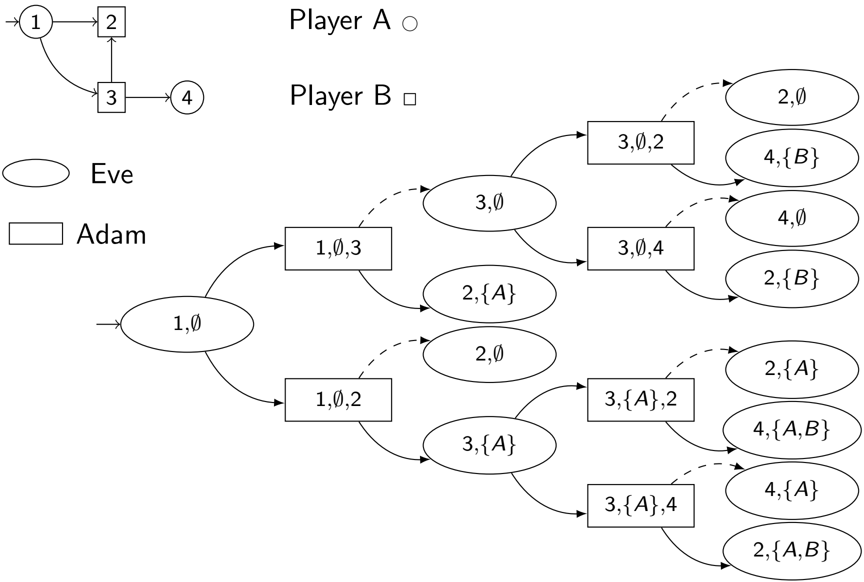


Figure: Example of a turn-based game

Player A ○
Player B □


1
2
3
4
Player A
Player B
Eve
Adam
1,∅
1,∅,3
1,∅,2
3,∅
2,{A}
2,∅
3,{A}
3,∅,2
3,∅,4
3,{A},2
3,{A},4
2,∅
4,{B}
4,∅
2,{B}
2,{A}
4,{A,B}
4,{A}
2,{A,B}

Notations and definitions

$$\pi_1((s, D)) = s \text{ and } \pi_2((s, D)) = D.$$

We extend these projections to plays, letting

$$\pi_1((s_0, D_0)(s_0, D_0, s_0')(s_1, D_1)\ldots) = s_0 s_1 \ldots$$

and

$$\pi_2((s_0, D_0)(s_0, D_0, s_0')(s_1, D_1)\ldots) = D_0 D_1 \ldots.$$

We define the set of deviators along a play η

$$L(\eta) = \lim_n \pi_2(\eta[2n]).$$

Eve's objective

Turn-based game $\mathcal{G}$ + a play ρ in $\mathcal{G} \rightsquigarrow$ the Deviator game $\mathcal{H}(\mathcal{G},\rho)$.

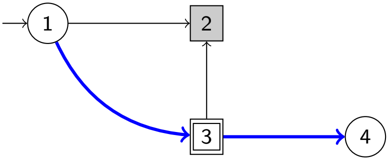
Eve's objective

Turn-based game $\mathcal{G}$ + a play ρ in $\mathcal{G} \rightsquigarrow$ the Deviator game $\mathcal{H}(\mathcal{G},\rho)$.

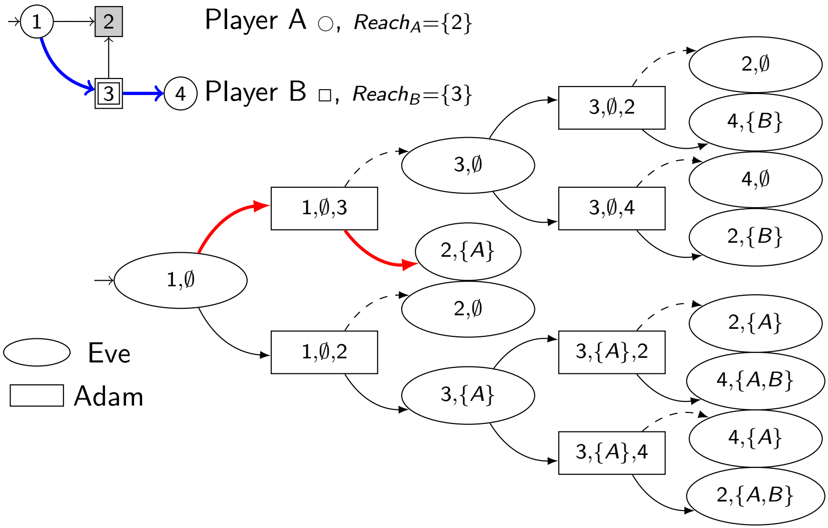
A play η in the zero-sum game $\mathcal{H}(\mathcal{G},\rho)$ is winning for Eve if

$$\forall C \supseteq L(\eta),\ \Big[|C| = 1 \Rightarrow \forall P \in C,\ \pi_1(\eta) \preccurlyeq_P \rho\Big].$$

Example



Player A $\bigcirc$, $Reach_A = \{2\}$
Player B $\square$, $Reach_B = \{3\}$
Consider $\rho = 134$. This play is only winning for player B.



$$\Omega_\exists = \{\eta \in \mathcal{H} \mid \forall C \supseteq L(\eta),\ |C| = 1 \Rightarrow \forall P \in C\ \pi_1(\eta) \preccurlyeq_P \rho\}$$

Romain's main theorem

A play η in the arena $\mathcal{H}(\mathcal{G}, \rho)$ is winning for Eve if

$$\forall C \supseteq L(\eta),\ \Big[|C| = 1 \Rightarrow \forall P \in C,\ \pi_1(\eta) \preccurlyeq_P \rho\Big].$$

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A play η in the arena $\mathcal{H}(\mathcal{G}, \rho)$ is winning for Eve if

$$\forall C \supseteq L(\eta),\ \Big[|C| = 1 \Rightarrow \forall P \in C,\ \pi_1(\eta) \preccurlyeq_P \rho\Big].$$

Theorem (R.Brenguier, 2012*)

There exists a Nash equilibrium in $\mathcal{G}$ whose outcome is ρ if and only if Eve has a winning strategy in $\mathcal{H}(\mathcal{G}, \rho)$ whose outcome η when Adam obeys is such that $\pi_1(\eta) = \rho$.

* Romain Brenguier, Nash equilibria in concurrent games - Application to timed games. Doctoral thesis, ENS Cachan, 2012.

Application

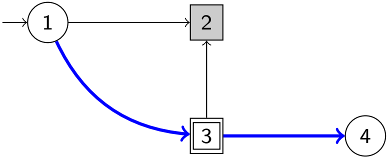


Figure: Turn-based game $\mathcal{G}$

Player A ○, $Reach_A = \{2\}$
Player B □, $Reach_B = \{3\}$
Consider $\rho = 134$. This play is only winning for player B.

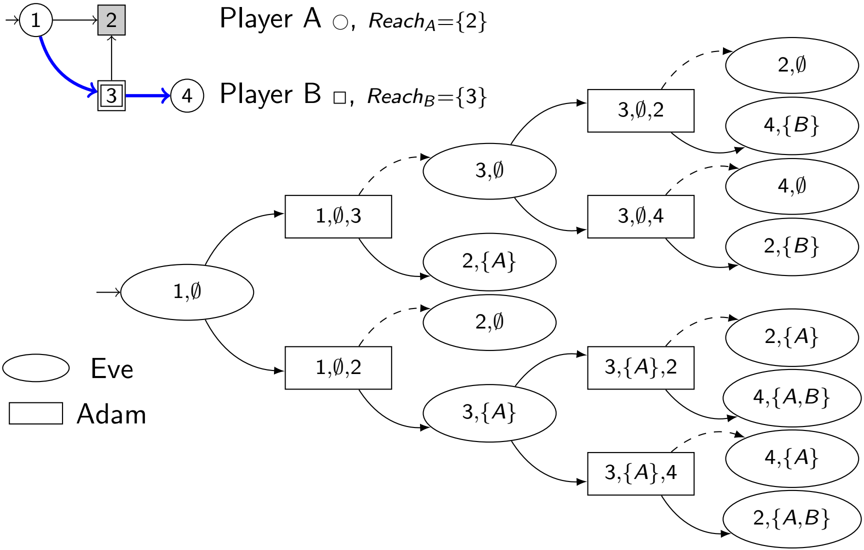


Figure: Deviator Game $\mathcal{H}(\mathcal{G}, \rho)$

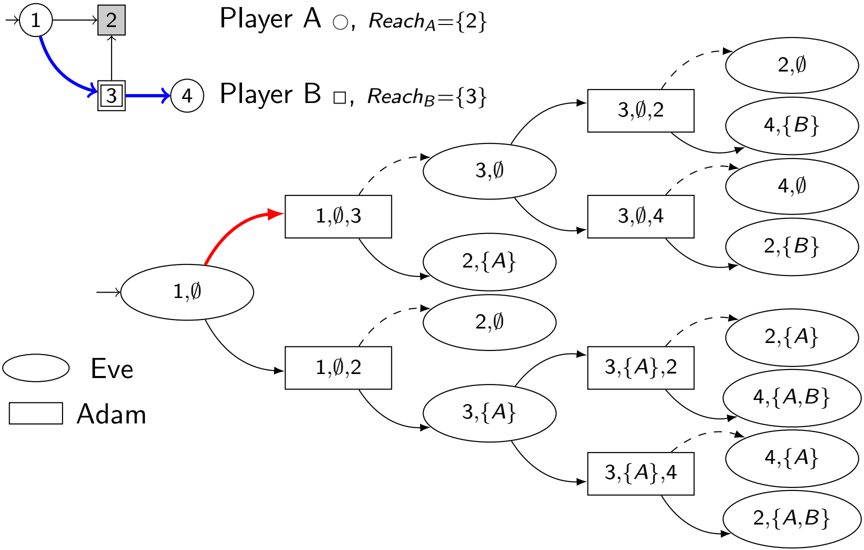
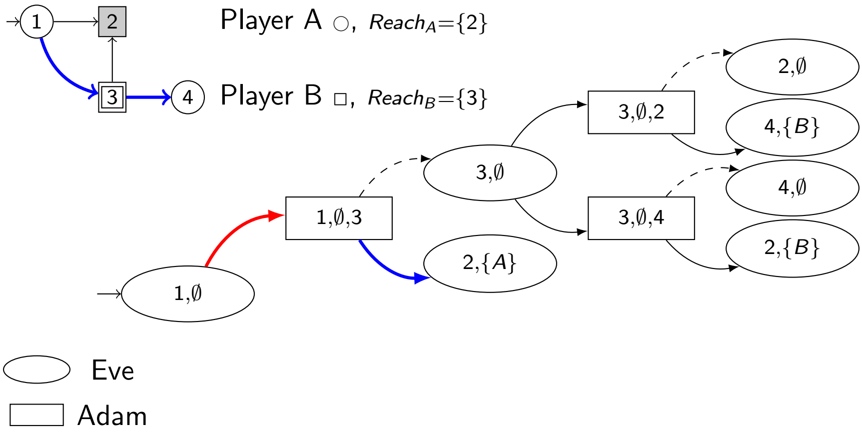


Figure: Deviator game $\mathcal{H}(\mathcal{G}, \rho)$



This path is not winning for Eve, thus there is no Nash equilibrium whose outcome is 134.

Warning

Unfortunately, this does not work for k-fault tolerant Nash equilibrium.

Warning

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$\longrightarrow$ Adaptation of Eve's winning objective

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Deviator Game

Input : Turn-based game $\mathcal{G}$ + a play ρ in $\mathcal{G}$ + β

The Deviator game $\mathcal{H}(\mathcal{G}, \rho)$ is a zero-sum turn-based game, played by Adam and Eve.

The game is played in the following way:

1. from a configuration (s, D), Eve chooses a successor v_1 of s;
2. the next state is (s, D, v_1);
3. then Adam chooses a successor v_2 of s;
4. Eve's new state is $(v_2, D \cup Dev((s, v_1), v_2))$.

$\longrightarrow$ Adaptation of Eve's winning objective

A play η in the arena $\mathcal{H}(\mathcal{G}, \rho)$ is winning for Eve if

$$\forall C \supseteq L(\eta),\ \Big[|C| = 1 \Rightarrow \forall P \in C,\ \pi_1(\eta) \preccurlyeq_P \rho\Big] \wedge \Big[|C| \leq k \Rightarrow \pi_1(\eta) \notin \quad \beta\Big].$$

$\longrightarrow$ Adaptation of Eve's winning objective

A play η in the arena $\mathcal{H}(\mathcal{G}, \rho)$ is winning for Eve if

$$\forall C \supseteq L(\eta),\ \left[|C| = 1 \Rightarrow \forall P \in C,\ \pi_1(\eta) \preccurlyeq_P \rho\right] \wedge \left[|C| \leq k \Rightarrow \pi_1(\eta) \notin \quad \beta\right].$$

Theorem

There exists a k-fault tolerant Nash equilibrium in $\mathcal{G}$ whose outcome is ρ if and only if Eve has a winning strategy in $\mathcal{H}(\mathcal{G}, \rho)$ whose outcome η when Adam obeys is such that $\pi_1(\eta) = \rho$.

Application

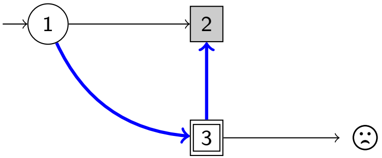


Figure: Turn-based game $\mathcal{G}$

Player A ○, $Reach_A = \{2\}$
Player B □, $Reach_B = \{3\}$
$\beta =$ ☹
Consider $\rho = 132$. This play is winning for both players.

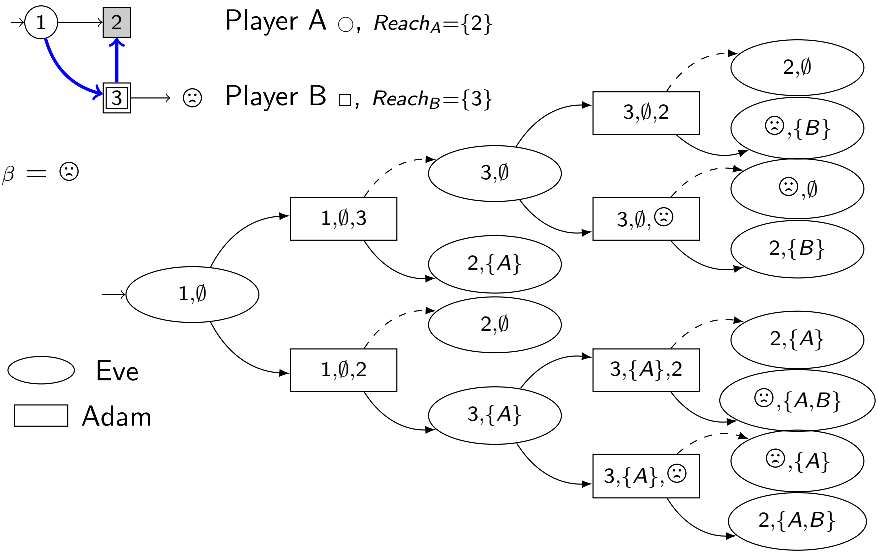
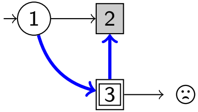


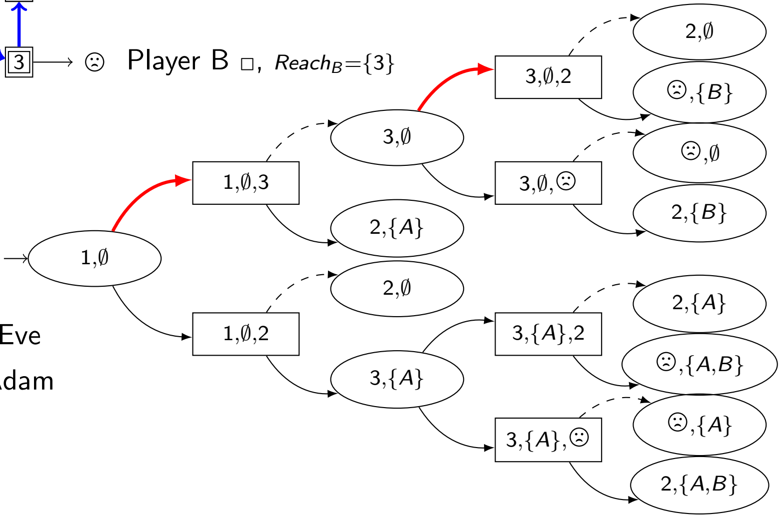
Figure: Deviator game $\mathcal{H}(\mathcal{G}, \rho)$



Player A ○, $Reach_A = \{2\}$

Player B □, $Reach_B = \{3\}$

$\beta = ☹$



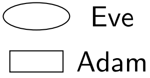
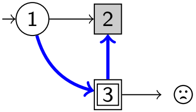


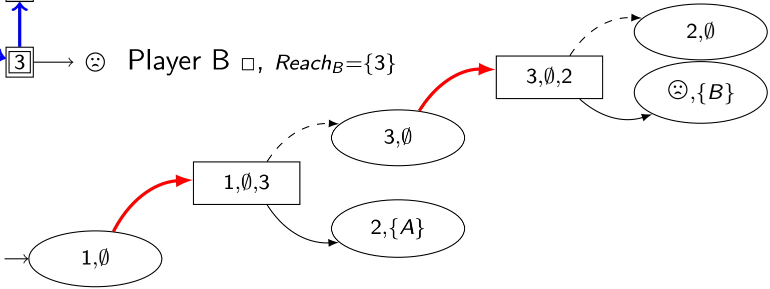
Figure: Deviator game $\mathcal{H}(\mathcal{G}, \rho)$

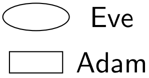


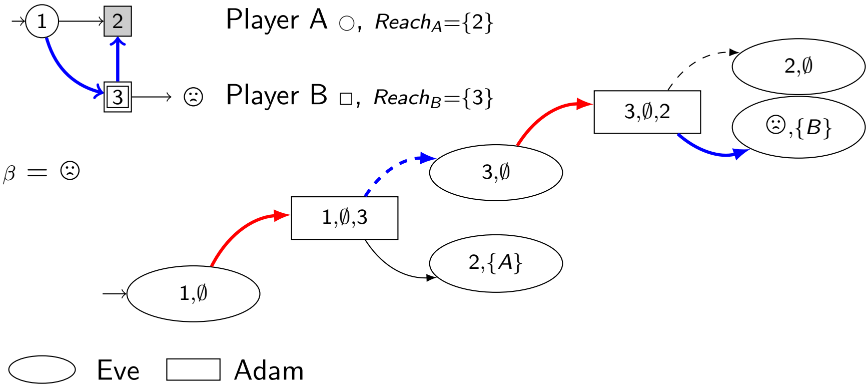
Player A ○, $Reach_A = \{2\}$

Player B □, $Reach_B = \{3\}$

β = ☹







$$\Omega_\exists = \Big\{\eta \in \mathcal{H} \mid \forall C \supseteq L(\eta),\ \Big[|C| = 1 \Rightarrow \forall P \in C,\ \pi_1(\eta) \preccurlyeq_P \rho\Big] \wedge \Big[|C| \leq k \Rightarrow \pi_1(\eta) \notin \beta\Big]\Big\}$$

This path is not winning for Eve, thus there is no 1-fault tolerant Nash equilibrium whose outcome is 132.

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Tool

Proposition (R. Brenguier, 2012*)

Let $\mathcal{G}$ be a turn-based game and s be the initial node. Assume that every player has a preference relation which only depends on the set of nodes that are visited and on the set of nodes that are visited infinitely often. If there is a Nash equilibrium with outcome ρ, then there is a Nash equilibrium with outcome ρ' of the form $\pi\tau^\omega$ such that $\rho \sim_A \rho'$ for all $A \in Agt$, and where $|\pi|$ and $|\tau|$ are bounded by $|V|^2$.

* Romain Brenguier, Nash equilibria in concurrent games - Application to timed games. Doctoral thesis, ENS Cachan, 2012.

Remark

The proposition still holds if we replace Nash equilibrium by k-fault tolerant Nash equilibrium.

Algorithm to find NE with reachability objectives with constraints

1. Guess a lasso-shaped play $\eta = \tau_1 \tau_2^{\omega}$ in $\mathcal{H}$ s.t $\pi_1(\eta)$ satisfies the constraints,
2. compute the set of winning states of Eve in $\mathcal{H}$,
3. check that η stays in this set.

$\longrightarrow$ Adaptation of this algorithm for 1-fault tolerant Nash equilibrium.

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Other results

Concurrent games $\longrightarrow$ Deviator Game
R. Brenguier, Robust equilibria in concurrent games, Arxiv, 2014

$\rightsquigarrow$ φ-equilibrium, an abstract concept that we use to work on concrete solution concepts.

Theorem

There exists a φ-equilibrium in $\mathcal{G}$ whose outcome is ρ if and only if Eve has a winning strategy in $\mathcal{H}(\mathcal{G}, \rho, \varphi)$ whose outcome η when Adam obeys is such that $\pi_1(\eta) = \rho$.

THANK YOU FOR YOUR ATTENTION !