

Secure Equilibria in Weighted Games

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Topic

Reactive systems :

- modeled by two-player zero-sum games
- the system against the environment
- synthesis of a controller = construction of a winning strategy

Several extensions :

- from Boolean objectives to quantitative objectives
- from two-player zero-sum games to multi-player non zero-sum games
- construction of an equilibrium

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Our work

- Two-player non zero-sum weighted games
- Measures : sup, inf, lim sup, lim inf, mean-payoff, and discounted sum
- Study of secure equilibria

Related work

- Secure equilibria introduced in [CHJ06] for qualitative objectives.
- Different kinds of equilibria (Nash, secure, subgame secure perfect, and subgame perfect) for quantitative reachability objectives [BBD10, BBDG12, KLST12].
- General results for the existence of Nash equilibria in a large class of weighted games [BDS13].
- Existence and constraint existence of Nash equilibria in weighted games for the mean-payoff measure [UW11].
- General hypotheses on multi-player weighted games that guarantee the existence of a secure equilibrium (not including the mean-payoff measure) [DFK⁺14].
- ...

1 Studied problems

2 Existence of secure equilibria

3 Constrained existence of secure equilibria

4 Uniqueness of maximal payoff profile of a secure equilibrium

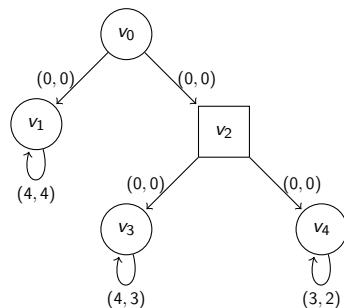
5 Conclusion and future work

Two player weighted games

Definition

$\mathcal{G} = (V, V_1, V_2, E, r, \text{Payoff}) :$

- (V, E) finite graph, with V_1, V_2 a partition of V ,
- $r = (r_1, r_2)$ weight function such that $r_i : E \rightarrow \mathbb{Q}$,
- $\text{Payoff} = (\text{Payoff}_1, \text{Payoff}_2)$ payoff function such that $\text{Payoff}_i : V^\omega \rightarrow \mathbb{R}$.



Strategy $\sigma : V^* V_i \rightarrow V$ for player i .

Finite-memory strategy. *Positional* strategy $\sigma : V_i \rightarrow V$.

Strategy profile (σ_1, σ_2) with *outcome* $\langle \sigma_1, \sigma_2 \rangle_{v_0}$ from initial vertex v_0 .

Payoff functions

The payoff function **Payoff** is one of the payoffs in $\{\text{Inf}, \text{Sup}, \text{LimInf}, \text{LimSup}, \text{InfMP}, \text{SupMP}, \text{Disc}^\lambda\}$ for $\lambda \in]0, 1[$, where for $\rho \in V^\omega$:

- $\text{Inf}_i(\rho) = \inf_{n \in \mathbb{N}} r_i(\rho_n, \rho_{n+1}),$
- $\text{LimInf}_i(\rho) = \liminf_{n \rightarrow \infty} r_i(\rho_n, \rho_{n+1}),$
- $\text{InfMP}_i(\rho) = \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} r_i(\rho_k, \rho_{k+1}),$
- $\text{Disc}_i^\lambda(\rho) = (1 - \lambda) \cdot \sum_{n=0}^{\infty} \lambda^n r_i(\rho_n, \rho_{n+1}),$

Similar definitions for $\text{Sup}_i(\rho)$, $\text{LimSup}_i(\rho)$, and $\text{SupMP}_i(\rho)$.

Equilibria

Non zero-sum game.

Objective of each player : maximize his own payoff.

Definition

The strategy profile (σ_1, σ_2) is a *Nash equilibrium* from vertex v_0 if, for each strategy σ'_i , $i \in \{1, 2\}$,

$$\text{Payoff}_1(\langle \sigma'_1, \sigma_2 \rangle_{v_0}) \leq \text{Payoff}_1(\langle \sigma_1, \sigma_2 \rangle_{v_0}),$$

$$\text{Payoff}_2(\langle \sigma_1, \sigma'_2 \rangle_{v_0}) \leq \text{Payoff}_2(\langle \sigma_1, \sigma_2 \rangle_{v_0}).$$

No player has an incentive to deviate from his strategy.

Equilibria

Objective of each player : first **maximize** his own payoff, and then **minimize** the payoff of the other player [CHJ06].

Definition

The strategy profile (σ_1, σ_2) is a **secure equilibrium** from vertex v_0 if, for each strategy $\sigma'_i, i \in \{1, 2\}$,

$$\text{Payoff}(\langle \sigma'_1, \sigma_2 \rangle_{v_0}) \preceq_1 \text{Payoff}(\langle \sigma_1, \sigma_2 \rangle_{v_0}),$$

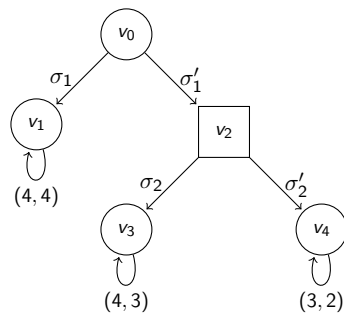
$$\text{Payoff}(\langle \sigma_1, \sigma'_2 \rangle_{v_0}) \preceq_2 \text{Payoff}(\langle \sigma_1, \sigma_2 \rangle_{v_0}).$$

No player has an incentive to deviate from his strategy w.r.t. his **lexicographic order**.

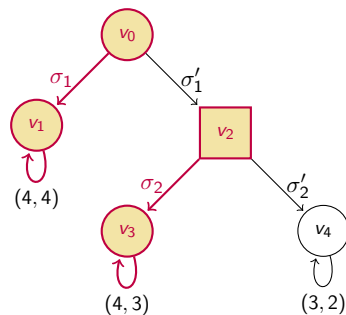
$$(x'_1, x'_2) \preceq_1 (x_1, x_2) \iff (x'_1 < x_1) \vee (x'_1 = x_1 \wedge x'_2 \geq x_2),$$

$$(x'_1, x'_2) \preceq_2 (x_1, x_2) \iff (x'_2 < x_2) \vee (x'_2 = x_2 \wedge x'_1 \geq x_1).$$

Example of an InfMP weighted game

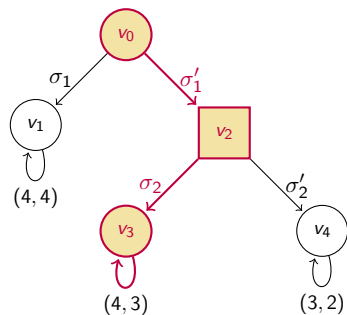


Example of an InfMP weighted game



- (σ_1, σ_2) : Nash equilibrium with payoff $(4, 4)$, but not a secure equilibrium since $(4, 4) \prec_1 (4, 3)$

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- (σ'_1, σ_2) : secure equilibrium with payoff $(4, 3)$

Studied problems

Let \mathcal{G} be a weighted game and v_0 be an initial vertex.

Problem

- 1 Does there *exist* a (finite-memory) secure equilibrium in \mathcal{G} from v_0 ?
- 2 What is the *complexity* of constructing a (finite-memory) secure equilibrium if one exists ?
- 3 Given two thresholds $\mu, \nu \in (\mathbb{Q} \cup \{\pm\infty\})^2$, is it decidable whether there exists a secure equilibrium (σ_1, σ_2) such that

$$\mu \leq \text{Payoff}(\langle \sigma_1, \sigma_2 \rangle_{v_0}) \leq \nu$$

i.e. $\mu_i \leq \text{Payoff}_i(\langle \sigma_1, \sigma_2 \rangle_{v_0}) \leq \nu_i$ for $i \in \{1, 2\}$?

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Existence

General approach to construct a finite-memory secure equilibrium.

We need two tools :

- lexicographic payoff games
- prefix-linear payoff functions

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Theorem

Let \mathcal{G} be a weighted game and v_0 an initial vertex. If for each $i \in \{1, 2\}$,

- \mathcal{G}^{\preceq_i} is *uniformly-determined*,
- Payoff _{i} is *prefix-linear*,

then there exists a finite-memory secure equilibrium in \mathcal{G} from v_0 with memory size bounded by $|V| + 2$.

Lexicographic payoff games

With the non zero-sum game $\mathcal{G} = (V, V_1, V_2, E, r, \text{Payoff})$, we associate two **zero-sum lexicographic payoff games**

$$\mathcal{G}^{\preceq_1} = (V, V_1, V_2, E, r, \text{Payoff}, \preceq_1),$$

$$\mathcal{G}^{\preceq_2} = (V, V_1, V_2, E, r, \text{Payoff}, \preceq_2).$$

In \mathcal{G}^{\preceq_1} , player 1 wants to **maximize** $\text{Payoff}(\rho)$ w.r.t. \preceq_1 , while player 2 wants to **minimize** it. Symmetrically for \mathcal{G}^{\preceq_2} .

Lexicographic payoff games

Let $\mathcal{G}^{\preceq 1}$ be a lexicographic payoff game.

Definition

- Let v be a vertex.

$$\overline{\text{Val}}(v) = \inf_{\sigma_2 \in \Sigma_2} \sup_{\sigma_1 \in \Sigma_1} \text{Payoff}(\langle \sigma_1, \sigma_2 \rangle_v),$$

$$\underline{\text{Val}}(v) = \sup_{\sigma_1 \in \Sigma_1} \inf_{\sigma_2 \in \Sigma_2} \text{Payoff}(\langle \sigma_1, \sigma_2 \rangle_v).$$

- $\mathcal{G}^{\preceq 1}$ is *determined* if, for every $v \in V$, we have $\overline{\text{Val}}(v) = \underline{\text{Val}}(v)$.
Value $\text{Val}(v)$ and optimal strategies.

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- $\mathcal{G}^{\preceq 1}$ is *positionally-determined* if it is determined and has positional optimal strategies for both players and all vertices.

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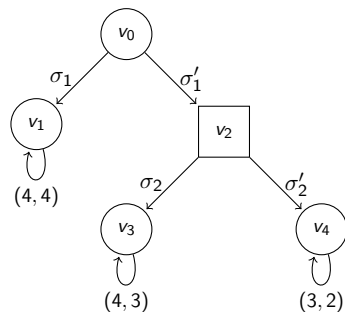
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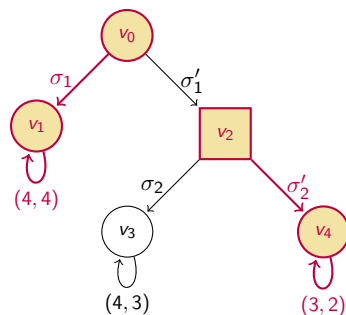
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- $\mathcal{G}^{\preceq 1}$ is *positionally-determined* if it is determined and has positional optimal strategies for both players and all vertices.
- $\mathcal{G}^{\preceq 1}$ is *uniformly-determined* if the positional optimal strategies can be chosen *globally* independently of the initial vertex v .

Example of an InfMP lexicographic payoff game



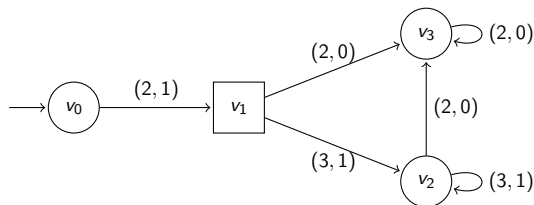
- lexicographic order \preceq_1 $(3, 2) \preceq_1 (4, 4) \preceq_1 (4, 3)$

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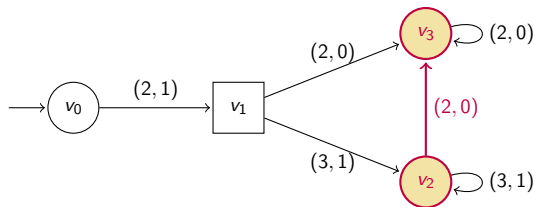
- lexicographic order \preceq_1 $(3, 2) \preceq_1 (4, 4) \preceq_1 (4, 3)$
- $\text{Val}(v_0) = (4, 4)$, $\text{Val}(v_2) = (3, 2)$

Example of an Inf lexicographic payoff game



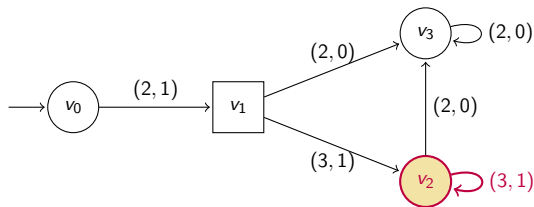
$$v_0 = v_1 = v_3 = (2, 0) \text{ and } v_2 = (3, 1)$$

Example of an Inf lexicographic payoff game



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Example of an Inf lexicographic payoff game



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Determinacy results

Theorem

Let $\mathcal{G}^{\preceq 1}$ be a lexicographic payoff game. Then $\mathcal{G}^{\preceq 1}$ is uniformly-determined (resp. positionally-determined) for payoff functions InfMP, SupMP, LimInf, LimSup and Disc^λ (resp. Inf and Sup).

Proof for

- **InfMP, SupMP** : inspired from [BCHJ09], however with several nontrivial adaptations ;
- **LimInf, LimSup, Inf, Sup** : based on adequate reductions to classical ω -regular games ;
- **Disc^λ** : based on one-dimensional discounted games.

Determinacy results (other interesting results)

Theorem

Let \mathcal{G}^{\preceq_1} be a lexicographic payoff game.

- 1 Let v be a vertex and $(\alpha, \beta) \in \mathbb{Q}^2$ be a pair of rationals. One can decide whether $(\alpha, \beta) \preceq_1 \text{Val}(v)$.
- 2 One can compute the (rational) *value* $\text{Val}(v)$ of each vertex v .
- 3 One can compute *optimal strategies* for both players in \mathcal{G}^{\preceq_1} .

	1.	2.	3.
InfMP, SupMP	$\text{NP} \cap \text{co-NP}$	<i>pseudo-polynomial</i>	<i>pseudo-polynomial</i>
LimInf, LimSup	P-complete	<i>polynomial</i>	<i>polynomial</i>
Inf, Sup	P-complete	<i>polynomial</i>	<i>polynomial</i>
Disc $^\lambda$	$\text{NP} \cap \text{co-NP}$	<i>pseudo-polynomial</i>	<i>pseudo-polynomial</i>

Prefix-linear payoff function

Definition

Function Payoff_i is *prefix-linear* if, $\forall h \in V^+$, $\exists a \in \mathbb{R}$, $\exists b \in \mathbb{R}^+$, $\forall \rho \in V^\omega$

$$\text{Payoff}_i(h\rho) = a + b \cdot \text{Payoff}_i(\rho).$$

In particular, it is *prefix-independent* if $\text{Payoff}_i(h\rho) = \text{Payoff}_i(\rho)$.

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- LimInf_i , LimSup_i , InfMP_i and SupMP_i : **prefix-independent**.
- Disc_i^λ : **not prefix-independent, but prefix-linear**.
- Inf_i , Sup_i : **not prefix-linear**.

Existence and construction

Theorem

Let \mathcal{G} be a weighted game and v_0 an initial vertex. If for each $i \in \{1, 2\}$,

- Payoff _{i} is *prefix-linear*,
- $\mathcal{G}^{\preceq i}$ is *uniformly-determined*,

then there exists a finite-memory secure equilibrium in \mathcal{G} from v_0 with memory size bounded by $|V| + 2$.

Moreover, if the uniform optimal strategies can be computed in C time for each $\mathcal{G}^{\preceq i}$, then this secure equilibrium can also be constructed in C time.

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Adaptation from [BDS13] : for **Nash equilibria** and **multiple** players.

Construction : each player i plays according to his optimal strategy in $\mathcal{G}^{\preceq i}$. As soon as player j deviates, the other player shifts to $\mathcal{G}^{\preceq j}$ and plays according to his optimal strategy in $\mathcal{G}^{\preceq j}$.

Existence and construction

Recall that for InfMP, SupMP, LimInf, LimSup, and Disc^λ payoffs,

- each Payoff_i is prefix-linear,
- each $\mathcal{G}^{\preceq i}$ is uniformly-determined.

Therefore we get :

Corollary

Existence of a finite-memory secure equilibrium in \mathcal{G} from v_0 .

	<i>Memory size</i>	<i>Construction time</i>
InfMP, SupMP	$ V + 2$	<i>pseudo-polynomial</i>
LimInf, LimSup	$ V + 2$	<i>polynomial</i>
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Inf, Sup	$ V \cdot E ^2 + 3$	<i>polynomial</i>
Disc^λ	$ V + 2$	<i>pseudo-polynomial</i>

- 1 Studied problems
- 2 Existence of secure equilibria
- 3 **Constrained existence of secure equilibria**
- 4 Uniqueness of maximal payoff profile of a secure equilibrium
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Constrained existence

Let $\mu, \nu \in (\mathbb{Q} \cup \{\pm\infty\})^2$ be two thresholds.

General approach to decide the existence of a secure equilibrium such that $\mu \leq \text{Payoff}(\langle \sigma_1, \sigma_2 \rangle_{v_0}) \leq \nu$, using the **two previous tools**

and a solution to the next problem :

Problem (*)

Let $G = (V, E, v_0, r, \text{Val})$ be a weighted graph with an initial vertex v_0 , and a value function $\text{Val} = (\text{Val}^1, \text{Val}^2)$, with $\text{Val}^i : V \rightarrow \mathbb{Q}$.

Is it decidable whether there exists an **infinite path** ρ from v_0 such that

- $\forall k \geq 0, \forall i \in \{1, 2\}, \text{Val}^i(\rho_k) \preceq_i \text{Payoff}(\rho_k \rho_{k+1} \dots)$, and
- $\mu \leq \text{Payoff}(\rho) \leq \nu$?

Constrained existence

Theorem

Let \mathcal{G} be a weighted game, and $\mu, \nu \in (\mathbb{Q} \cup \{\pm\infty\})^2$ two thresholds. If

- for each $i \in \{1, 2\}$, Payoff_i is prefix-linear, and $\mathcal{G}^{\preceq i}$ is uniformly-determined with computable values,
- Problem (*) is decidable for the graph G constructed from \mathcal{G} such that both functions Val^i , $i \in \{1, 2\}$, are defined from the values $\text{Val}^i(v)$, $v \in V$, in $\mathcal{G}^{\preceq i}$.

One can decide whether there exists a secure equilibrium (σ_1, σ_2) in \mathcal{G} from an initial vertex v_0 such that $\mu \leq \text{Payoff}(\langle \sigma_1, \sigma_2 \rangle_{v_0}) \leq \nu$.

Lemma

Let ρ be a play in \mathcal{G} from v_0 . Then ρ is the **outcome of a secure equilibrium** iff $\forall k \geq 0, \forall i \in \{1, 2\}, \text{Val}^i(\rho_k) \preceq_i \text{Payoff}(\rho_k \rho_{k+1} \dots)$.

Solution to Problem (*)

Theorem

Problem () is decidable in polynomial time for InfMP, SupMP, LimInf and LimSup payoffs.*

Corollary

Let \mathcal{G} be a weighted game and v_0 be an initial vertex. One can decide whether there exists a secure equilibrium (σ_1, σ_2) such that $\mu \leq \text{Payoff}(\langle \sigma_1, \sigma_2 \rangle_{v_0}) \leq \nu$.

	<i>Complexity</i>
InfMP, SupMP	$\text{NP} \cap \text{co-NP}$
LimInf, LimSup	P

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	<i>Complexity</i>
InfMP, SupMP	NP \cap co-NP
LimInf, LimSup	P
Inf, Sup	P
Disc ^{λ}	open

Constrained existence

The case of Disc^λ payoff is left **open**. Indeed the next problem [CFW13] reduces to the constrained existence of a secure equilibrium for Disc^λ payoff.

Problem (**)

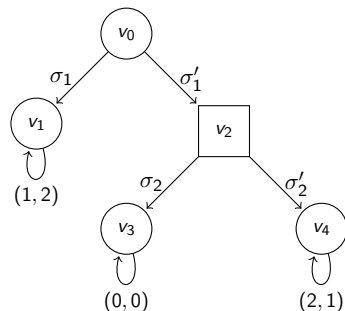
Given three rational numbers a, b and t , and a rational discount factor $\lambda \in]0, 1[$, does there exist an infinite sequence $w = w_0 w_1 \dots \in \{a, b\}^\omega$ such that $\sum_{k=0}^{\infty} w_k \lambda^k$ is equal to t ?

Problem (**) is related to other hard open problems in diverse mathematical fields [BO14].

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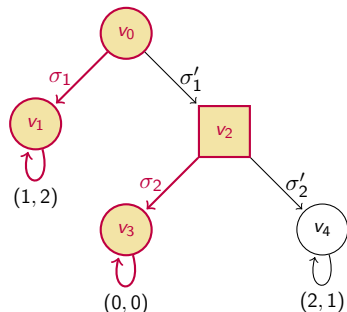
Counterexample

An InfMP weighted game with no unique maximal payoff profile of a secure equilibrium



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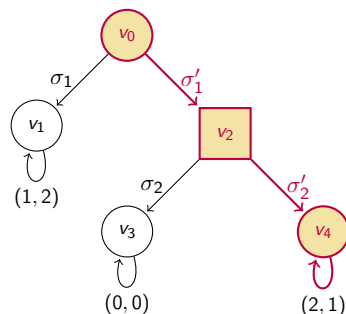
An InfMP weighted game with no unique maximal payoff profile of a secure equilibrium



- (σ_1, σ_2) : secure equilibrium with payoff $(1, 2)$

Counterexample

An InfMP weighted game with no unique maximal payoff profile of a secure equilibrium



- (σ_1, σ_2) : secure equilibrium with payoff $(1, 2)$
- (σ'_1, σ'_2) : secure equilibrium with payoff $(2, 1)$

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Conclusion

- **General hypotheses** for the existence and constrained existence of secure equilibria in two-player weighted games
 - **as corollary**, solutions for InfMP, SupMP, LimInf, LimSup, Inf, Sup and Disc^λ payoffs
 - **except** for the constrained existence for Disc^λ payoff
- **Interesting results** about the zero-sum games $\mathcal{G}^{\preceq 1}$, $\mathcal{G}^{\preceq 2}$
- **No uniqueness** of maximal payoff profile of a secure equilibrium

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- **No uniqueness** of maximal payoff profile of a secure equilibrium

Future work

- Secure equilibria in **multi-player** weighted games?
- If several secure equilibria exist, selection of an **interesting** one (Pareto optimal, ...)?
- **Subgame perfect** and **secure subgame perfect** equilibria in weighted games?
- **More than one payoff** for each player?



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