Cassting

Symmetry reduction for games

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SEVENTH FRAMEWORK PROGRAMME

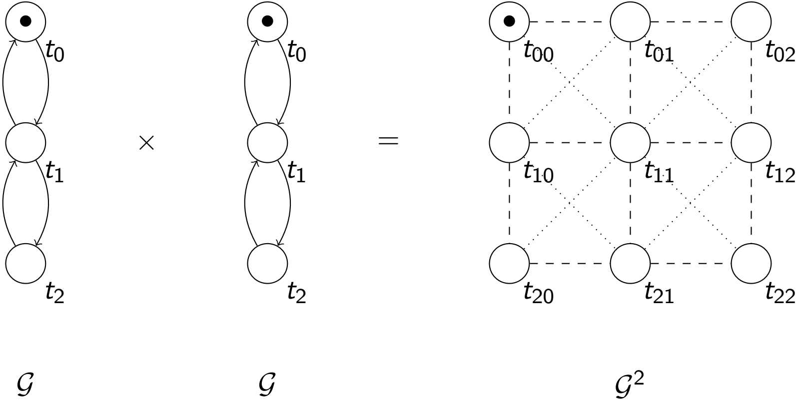
Motivation

Practical instances of games for synthesis can

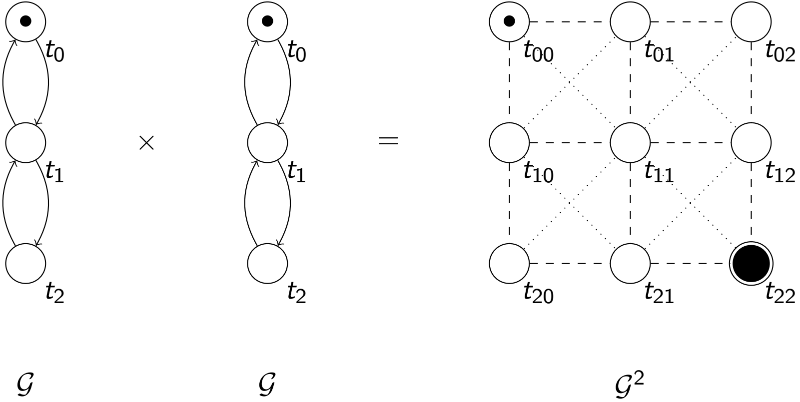
- have (very) **large state spaces**
- exhibit **symmetric behavior** (equivalent processes, subcomponents, etc.)

Goal: **Reduce state space** by using **symmetry** without building the original state space

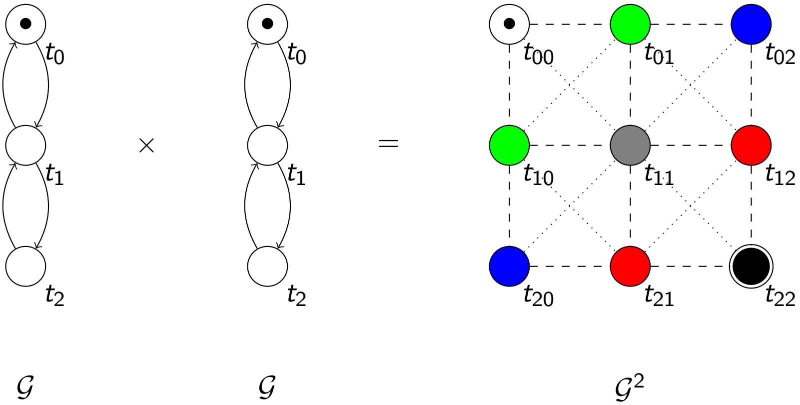
Example - Product game



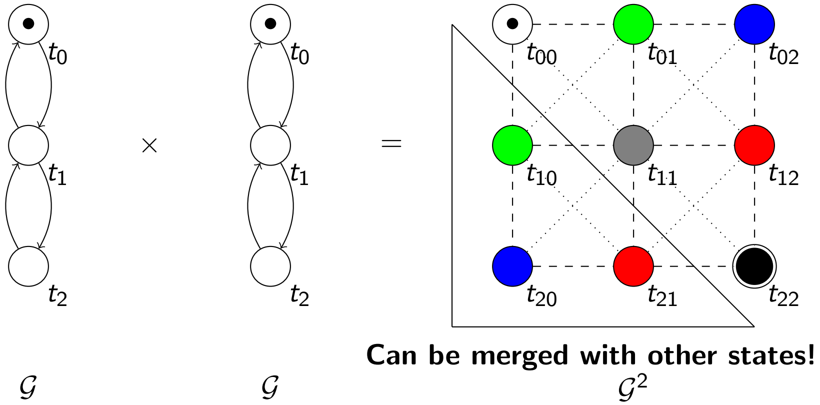
Example - Product game



Example - Product game



Example - Product game



Model-checking

Similar problems have been considering in **model-checking** of:

- **Transition systems** [Clarke et al. '96, Emerson et al. '96, Ip, Dill '96]
- **Timed systems** [Hendriks et al. '03]
- **Probabilistic systems** [Kwiatkowska et al. '06]

Reductions are used in practice

Two-player turn-based games

Given a **turn-based game** $\mathcal{G} = (S, R, S_1, S_2)$

- An **outcome** is a sequence $s_0 s_1 \ldots \in S^\omega$ so $(s_i, s_{i+1}) \in R$
- A **history** is a finite prefix of an outcome
- A **strategy** for player i is a mapping $\sigma : \text{Hist} \to S$ defined on all histories ending in S_i satisfying $(s_k, \sigma(s_0 \ldots s_k)) \in R$
- An **objective** is a subset of the set of outcomes

Symmetry groups

A **symmetry** is a permutation π of S such that

- $(s, s') \in R \Leftrightarrow (\pi(s), \pi(s')) \in R$
- $s \in S_1 \Leftrightarrow \pi(s) \in S_1$

A **symmetry group** is a set G of permutations such that $(G, \circ)$ is a group.
The **orbit** of a state $s \in S$:

$$\theta(s) = \{s' \in S \mid \exists \pi \in G . \pi(s) = s'\}$$

The orbits induce an equivalence relation $\sim$ on S such that $s \sim s'$ iff $s \in \theta(s')$

Quotient game

Given a TB game $\mathcal{G} = (S, R, S_1, S_2)$ and a symmetry group G define the **quotient game** $M = (S_M, R_M, S_{M1}, S_{M2})$ where

- $S_M = \{\theta(s) \mid s \in S\}$
- $R_M = \{(\theta(s), \theta(s')) \mid (s, s') \in R\}$
- $S_{Mi} = \{\theta(s) \mid s \in S_i\}$ for $i = 1, 2$

An outcome $s_0 s_1 \ldots$ in $\mathcal{G}$ **corresponds to** an outcome $t_0 t_1 \ldots$ in M if $s_i \in t_i$ for all $i \geq 0$.

Correspondence lemma

Lemma

Let $\mathcal{G}$ be a TB game, G be a symmetry group and $s_0 \in S$. Then

1. **For each** *player 1 strategy σ in $\mathcal{G}$* **there exists** *a player 1 strategy σ' in M* **so every** $\beta \in \mathrm{Out}_M(\theta(s_0), \sigma')$ *corresponds to some* $\alpha \in \mathrm{Out}_{\mathcal{G}}(s_0, \sigma)$
2. **For each** *player 1 strategy σ' in M* **there exists** *a player 1 strategy σ in $\mathcal{G}$* **so every** $\alpha \in \mathrm{Out}_{\mathcal{G}}(s_0, \sigma)$ *corresponds to some* $\beta \in \mathrm{Out}_M(\theta(s_0), \sigma')$

Preservation of objectives

A symmetry group G **preserves objective** Ω in the TB game $\mathcal{G}$ if for all outcomes ρ, ρ'

$$\rho \in \Omega \wedge \rho \sim \rho' \Rightarrow \rho' \in \Omega$$

If G preserves Ω, define $\Omega_G = \{\theta(s_0)\theta(s_1)\ldots \mid s_0 s_1 \ldots \in \Omega\}$

Theorem

If G **preserves** *Ω, then*

- *Player 1 has* **a winning strategy** *from s in $\mathcal{G}$ with objective Ω if and only if*
- *Player 1 has* **a winning strategy** *from $\theta(s)$ in M with objective Ω_G*

Parity games

A **coloring** is map $c : S \to \{1, ..., k\}$
A symmetry group G **preserves** c if $s \sim s' \Rightarrow c(s) = c(s')$.

Proposition

If G preserves c then

1. *G preserves the objective*
   $\Omega = \{s_0 s_1 ... \in \mathrm{Out}_{\mathcal{G}} \mid \min_{i \geq 0}(c(s_i)) \equiv 1 (\mathrm{mod}\ 2)\}$
2. $\Omega_G = \{\theta(s_0)\theta(s_1)... \in \mathrm{Out}_M \mid \min_{i \geq 0}(c(\theta(s_i))) \equiv 1 (\mathrm{mod}\ 2)\}$
3. *Player 1* **is winning** *in $\mathcal{G}$ from state s with objective Ω iff Player 1* **is winning** *in M from state $\theta(s)$ with objective Ω_G*

Alternating-time temporal logic

A **labeling** is map $L : S \to 2^{\text{PROP}}$ for a finite set PROP of proposition symbols.
A symmetry group G **preserves** L if $s \sim s' \Rightarrow L(s) = L(s')$.

Proposition

If G preserves L then for all ATL* formulae φ and $s \in S$

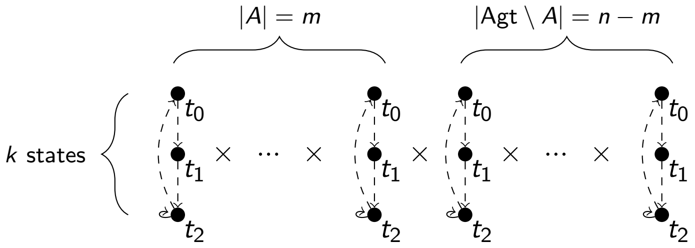
$$(\mathcal{G}, L), s \models \varphi \text{ if and only if } (M, L), \theta(s) \models \varphi$$

Concurrent games

All results hold for **concurrent games** as well

However, symmetries must also map **actions to actions**

Product game revisited



Parikh objective: Only depends on the number of agents from A and $\mathrm{Agt} \setminus A$ in each local state.
Examples of Parikh objectives:

- Always at least as many agents from A as from $\mathrm{Agt} \setminus A$ in t_1
- Never more than one agent from $\mathrm{Agt} \setminus A$ in t_1

Product game revisited

We can **define a symmetry group** G preserving Parikh objectives.
Orbits are pairs of Parikh images.
Original game has $|S| = k^n$ states. Quotient game has

$$|S_M| = \binom{n-m+k-1}{k} \cdot \binom{m+k-1}{k}$$

$$\leq \frac{(n-m+k-1)^k}{k!} \cdot \frac{(m+k-1)^k}{k!}$$

For a fixed k this is polynomial in n (and m).

Pros and cons

Pros:

- Games **do not have to be symmetric** in all aspects
- **Independent of algorithm** used to solve a game
- Test results from model-checking are **promising**

Cons:

- Not all games posses a lot of symmetry
- The engineer has to **provide** the symmetry

Extensions

It would be interesting to try to extend results to

- **Stochastic** games/strategies
- **Incomplete information** games
- **Timed** games
- **Equilibrium** computation

Question

Questions?