# Simple strategies for Banach-Mazur games and fairly correct systems

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Second Cassting meeting – Aalborg

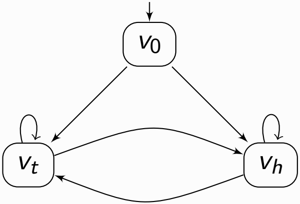
# Outline of the talk

# Motivations

## Classical Model-Checking

Given a model $M$ and a property $\varphi$, decide whether :

$$M \models \varphi, \quad \text{i.e. } \{\rho \text{ execution of } M \mid \rho \not\models \varphi\} \text{ is } \textbf{empty}.$$




$M_{\text{coin}} \not\models \mathbf{F}\ \text{head} \quad ; \quad M_{\text{coin}} \not\models \mathbf{GF}\ \text{tails}$

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## Fair Model-Checking

Given a model $M$ and a property $\varphi$, decide whether :

$$M \mathrel{|\!\!\approx} \varphi, \quad \text{i.e. } \{\rho \text{ execution of } M \mid \rho \not\models \varphi\} \text{ is } \textbf{“very small”}.$$

How to formalise the **fair model-checking** ?

- Via probability

$$\begin{aligned} M \mathrel{|\!\!\approx}_{\mathbb{P}} \varphi \quad & \textit{iff} \quad \mathbb{P}(\{\rho \text{ of } M \mid \rho \not\models \varphi\}) = 0 \\ & \textit{iff} \quad \mathbb{P}(\{\rho \text{ of } M \mid \rho \models \varphi\}) = 1 \end{aligned}$$

- Via topology

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Given a model $M$ and property $\varphi$, do we have that

$$M \mathrel{|\!\!\approx_{\mathbb{P}}} \varphi \quad \Leftrightarrow \quad M \mathrel{|\!\!\approx_{T}} \varphi \quad ???$$

In general, the answer is **NO** (example later).

## Theorem [VV06]

Given a finite system $M$ and an $\omega$-regular property $\varphi$, we have that

$$M \mathrel{|\!\!\approx_{\mathbb{P}}} \varphi \quad \Leftrightarrow \quad M \mathrel{|\!\!\approx_{T}} \varphi,$$

for bounded Borel measures.

Can we go further ?

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1. Motivations
2. Banach-Mazur games and large sets
3. Simple strategies for Banach-Mazur games
4. The concept of $\alpha$-strategy

# Banach-Mazur games

## Definition

A *Banach-Mazur game* $\mathcal{G}$ on a finite graph is a triplet $(G, v_0, W)$ where

- $G = (V, E)$ is a finite directed graph with no deadlock,
- $v_0 \in V$ is the initial state,
- $W \subset V^\omega$.

Given $(G, v_0, W)$, Pl. 0 and Pl. 1 play as follows :

- Pl. 1 begins with choosing a finite path $\rho_1$ starting in $v_0$ ;
- Pl. 0 prolongs $\rho_1$ by choosing another finite path $\rho_2$ ;
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# Strategies for Banach-Mazur games

Let $G = (V, E)$ be a graph and $\mathcal{G} = (G, v_0, W)$ be a Banach-Mazur game.

**Definition**

A strategy for Pl. 0 is a function $f : V^* \to V^*$ such that whenever

$$f(\rho) = \rho'$$

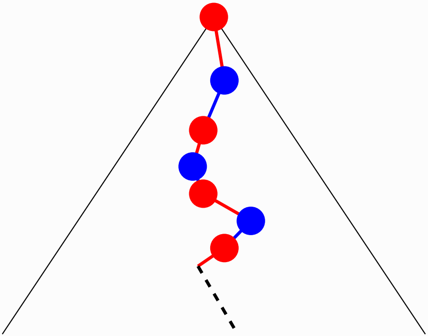
we have that $\rho'$ prolongs $\rho$ in $G$.

$\rho_1 \rho_2 \rho_3 \rho_4 \rho_5 \rho_6 \rho_7 \cdots$

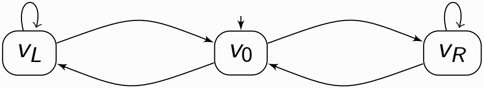
$f(\rho_1) = \rho_2$

$f(\rho_1 \rho_2 \rho_3) = \rho_4$

$f(\rho_1 \rho_2 \rho_3 \rho_4 \rho_5) = \rho_6$



# Banach-Mazur game : an example

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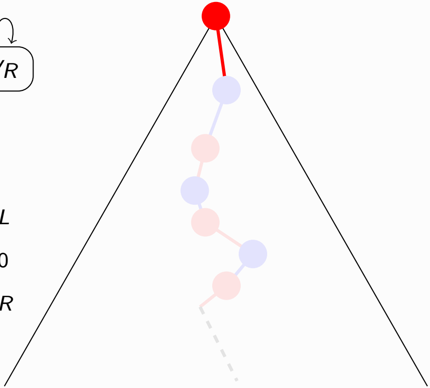

$f$ is a winning strategy for Pl. 0 !!!

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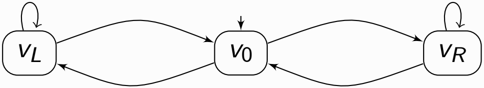


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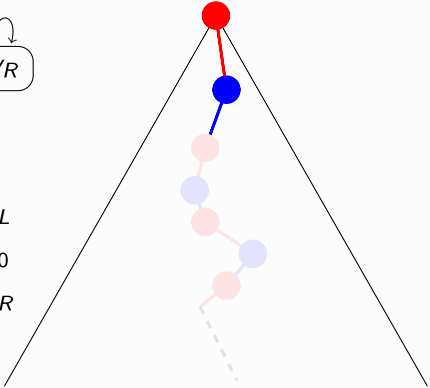

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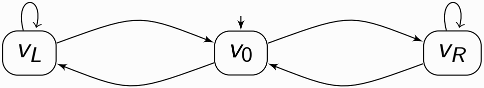


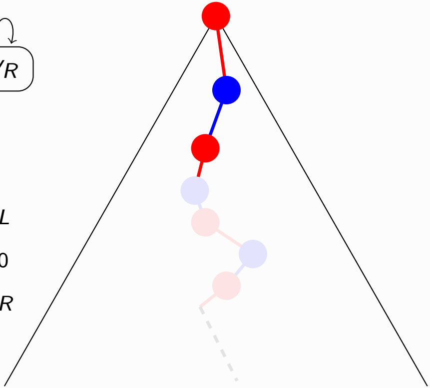

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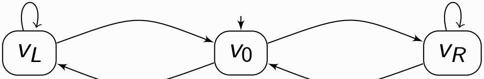


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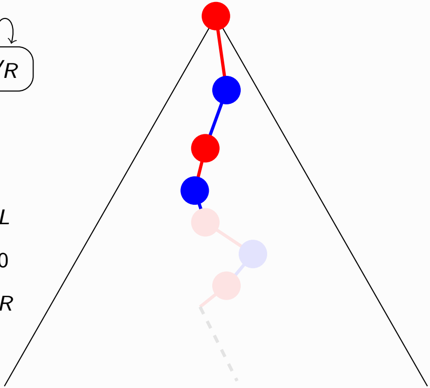

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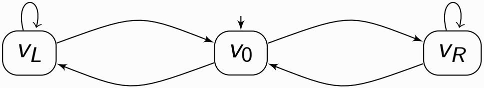


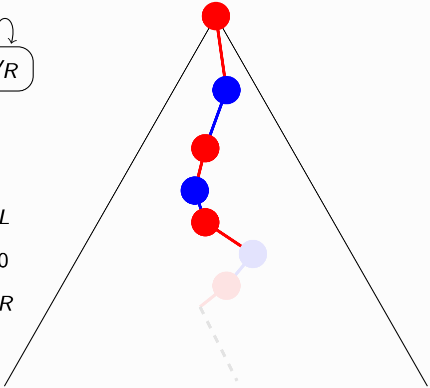

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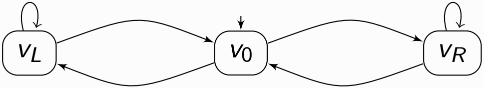


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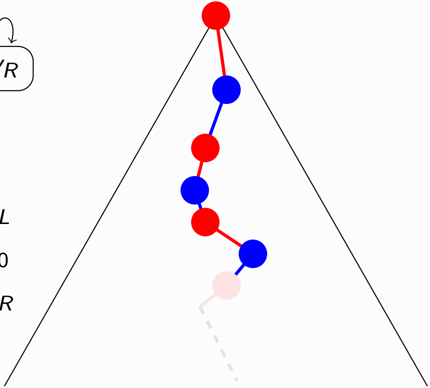

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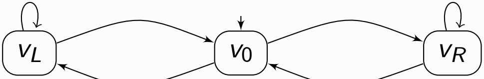


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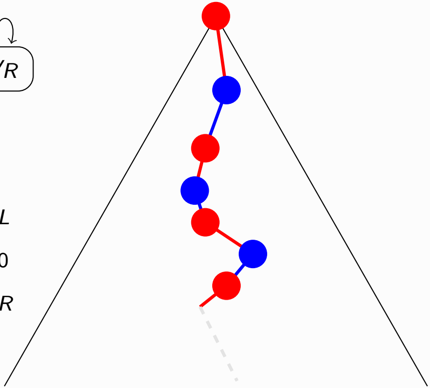

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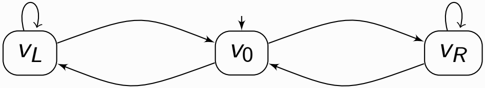


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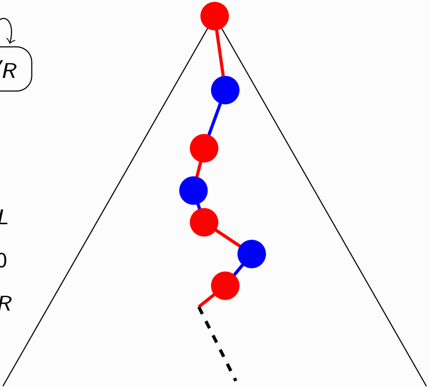

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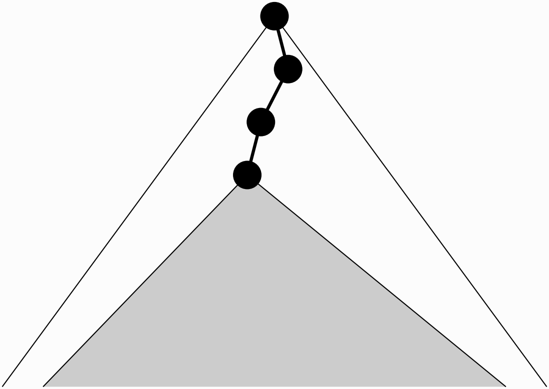
# The Cantor topology

Given $V$ a finite set, let $(a_i)_{i\in\mathbb{N}}$ and $(b_i)_{i\in\mathbb{N}}$ be two elements of $V^\omega$.

$$d((a_i)_{i\in\mathbb{N}},(b_i)_{i\in\mathbb{N}}) = 2^{-k} \quad \text{where} \quad k = \min\{i \in \mathbb{N} \mid a_i \neq b_i\}.$$

The metric induces a topology whose basic open sets are the open balls



# Banach-Mazur games and large sets

Let $(V, E)$ be a graph, where $V^\omega$ equipped with the Cantor topology.

**Definitions**

A set $S \subseteq V^\omega$ is said

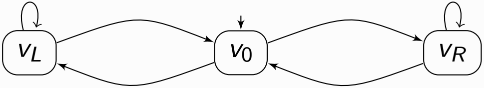
- *nowhere dense* if the closure of $S$ has empty interior.
- *meagre* if it can be seen as a countable union of nowhere dense sets.
- large if $S^c$ is meagre.

**Theorem [Oxtoby57]**

Let $\mathcal{G} = (G, v_0, W)$ be a Banach-Mazur game on a finite graph.

Pl. 0 has a winning strategy for $\mathcal{G}$ if and only if $W$ is large.

# Banach-Mazur game : an example

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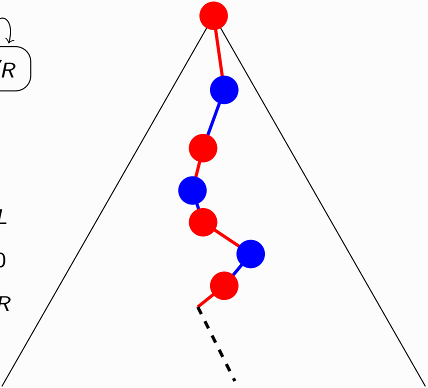
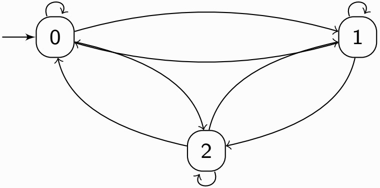


The set $W$ is thus a large set ! ! !

# A large set of probability 0

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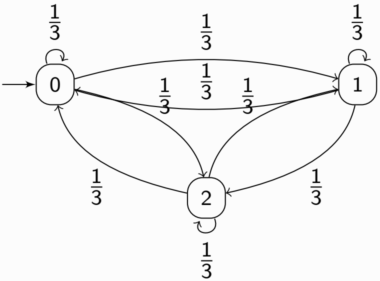


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$\rightsquigarrow$ $W$ is large.

$$\mathbb{P}(W) \leqslant \sum_{n=1}^{\infty} \mathbb{P}(\{w \in W \mid \text{ the first palyndrom has length } 2n\})$$

$$= \sum_{n=1}^{\infty} \mathbb{P}(\{w \in \{0,1,2\}^\omega \mid \text{ the first palyndrom has length } 2n\}) \cdot \mathbb{P}(W)$$

$$\leqslant \sum_{n=1}^{\infty} \frac{\mathbb{P}(W)}{3^n} = \frac{\mathbb{P}(W)}{2}$$

$\rightsquigarrow$ $\mathbb{P}(W) = 0$ !!!

# Outline of the talk

1. Motivations
2. Banach-Mazur games and large sets
3. Simple strategies for Banach-Mazur games
4. The concept of $\alpha$-strategy

# Back to our motivation

**Theorem [VV06]**

Given a finite system $M$ and an $\omega$-regular property $\varphi$, we have that

$$M \approx\!\!\!\models_{\mathbb{P}} \varphi \quad \Leftrightarrow \quad M \approx\!\!\!\models_{T} \varphi,$$

for bounded Borel measures.

The key ingredient to prove the above result is the following result :

**Theorem [BGK03]**

Given $\mathcal{G} = (G, v_0, W)$ where $W$ is an $\omega$-regular property, we have that

Pl. 0 has a winning strategy for $\mathcal{G}$
iff
Pl. 0 has a **positional** winning strategies for $\mathcal{G}$.

# Simple strategies for Banach-Mazur game

Given $\mathcal{G} = (G, v_0, W)$, let $f : V^* \to V^*$ be a strategy for Pl. 0.

$$\underbrace{f(\rho_1\rho_2\cdots\rho_{2n+1})}_{\text{What is observed}} \quad = \quad \underbrace{\rho_{2n+2}}_{\text{What is played}}$$

We say that $f$ is

- *positional* if it only depends on Last($\rho_{2n+1}$).
- *finite memory* if it only depends on Last($\rho_{2n+1}$) and a finite memory.
- *b-bounded* if $|\rho_{2n+2}| \leqslant b$.
- *bounded* if there is $b \geqslant 1$ such that $f$ is b-bounded.
- *move-counting* if it only depends on Last($\rho_{2n+1}$) and the number of moves already played.
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E. Grädel, S. Leßenich, *Banach-Mazur Games with Simple Winning Strategies*, CSL 2012

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# About Simple strategies for Pl. 0

**Theorem [BGK03]**

Given $\mathcal{G} = (G, v_0, W)$ on a finite graph, we have that

Pl. 0 has a **positional** winning strategy for $\mathcal{G}$
iff
Pl. 0 has a **finite-memory** winning strategies for $\mathcal{G}$.

**Simple observation**

Given $\mathcal{G} = (G, v_0, W)$ on a finite graph, we have that

If Pl. 0 has a **positional** winning strategy for $\mathcal{G}$,
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# About Simple strategies for Pl. 0 (cntd)

**Theorem [BM13]**

Given $\mathcal{G} = (G, v_0, W)$ on a finite graph, we have that

Pl. 0 has a **length-counting** winning strategy for $\mathcal{G}$

iff

Pl. 0 has a winning strategies for $\mathcal{G}$.

# Sketck of proof

Let $f$ be a winning strategy for Pl. 0
We have to build a length-counting winning strategy, i.e.

$$h : V \times \mathbb{N} \to V^*$$

Let $\{\pi_1, \ldots, \pi_m\}$ be the set finite paths of length $n$ such ending in $v$.

We then let

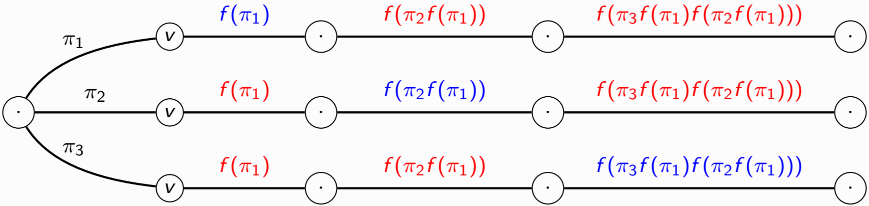
$$h(v,n) = f\left(\pi_1\right) f\left(\pi_2 f(\pi_1)\right) f\left(\pi_3 f(\pi_1) f(\pi_2 f(\pi_1))\right) \ldots \\ \cdots f\left(\pi_m f(\pi_1) f(\pi_2 f(\pi_1)) \cdots\right).$$

We prove that $h$ is a length-counting winning strategy for Pl. 0.

# Sketck of proof (ctnd)

Example where $m = 3$, we have $\{\pi_1, \pi_2, \pi_3\}$, and we then let

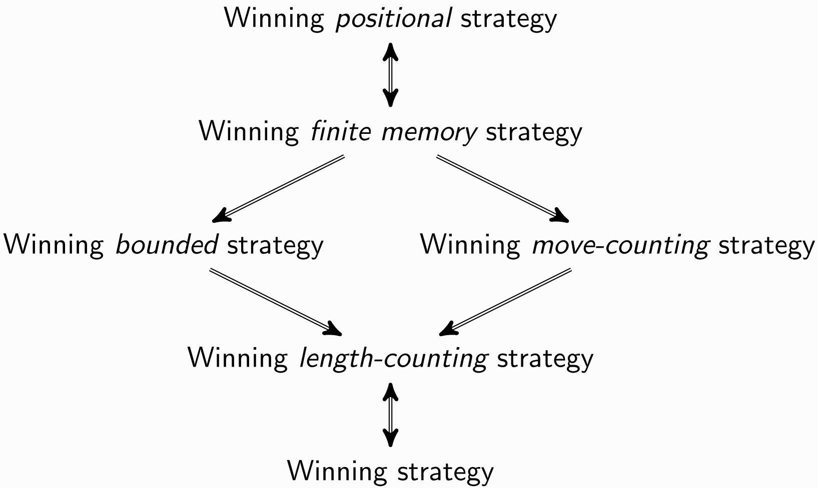
$$h(v, n) = f(\pi_1) f(\pi_2 f(\pi_1)) f(\pi_3 f(\pi_1) f(\pi_2 f(\pi_1)))$$




If $\rho$ is consistent with $h$, then $\rho$ is consistent with $f$ (which is winning).

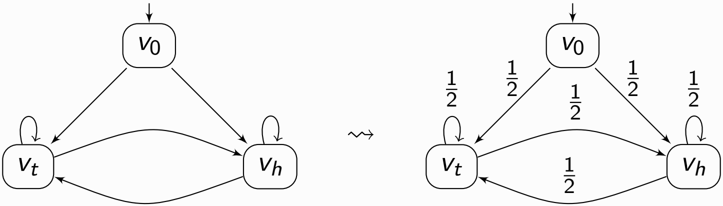
$\rightsquigarrow$ $h$ is a length-counting winning strategy for Pl. 0.

# Simple strategies for Pl. 0 on finite graphs




Combining simple observations and results from [BGK03], [VV06], [GL12], [BM13]

# Relations with the sets of probability one




**Proposition**

Let $\mathcal{G} = (G, v_0, W)$ be a Banach-Mazur game on a finite graph and $\mathbb{P}$ a reasonable probability measure.

If Pl. 0 has $\begin{cases} \text{a move-counting} \\ \text{a bounded} \end{cases}$ winning strategy for $\mathcal{G}$, then $\mathbb{P}(W) = 1$.

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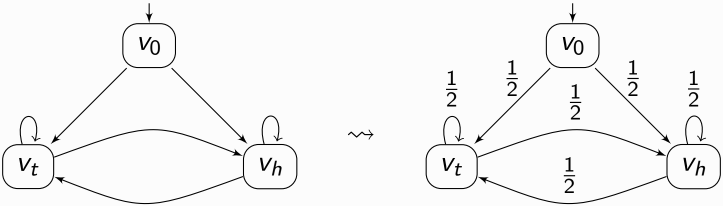


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There exist large **open** set of probability 1 without a positional/ bounded/ move-counting winning strategy.

$$W = \{(w_k)_{k \geqslant 1} \in \{0,1\}^\omega \mid \exists n > 1 \; w_{n!} = 1\}$$

We look for a new concept of "simple strategy"

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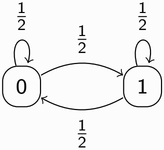
# Back to the example




$$W = \{(w_k)_{k \geqslant 1} \in \{0,1\}^\omega \mid \exists n > 1 \; w_{n!} = 1\}$$

Clearly Pl. 0 has a winning strategy, thus $W$ is **large**.

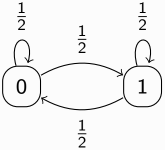
Moreover, we have that $\mathbb{P}(W) = 1$. Indeed, for $n > 1$ :

$$A_n := \{(w_k)_{k \geqslant 1} \in \{0,1\}^\omega \mid w_{n!} = 1 \text{ and } w_{m!} = 0 \text{ for any } 1 < m < n\},$$

we thus have :

$$W = \dot{\bigcup_{n>1}} A_n \quad \text{and} \quad \mathbb{P}(A_n) = \frac{1}{2^{n-1}} \qquad \rightsquigarrow \qquad \mathbb{P}(W) = 1.$$

# Back to the example




$$W = \{(w_k)_{k \geqslant 1} \in \{0,1\}^\omega \mid \exists n > 1 \ \ w_{n!} = 1\}$$

Let $f$ be a $b$-**bounded** strategy for Pl. 0.

A winning strategy for Pl. 1 consists in

- starting by playing $(b+1)!$ zeros,
- at each step, completing the sequence by 0's to reach the next $k!$

$\leadsto$ there is no winning bounded (resp. positional) strategy for Pl. 0.

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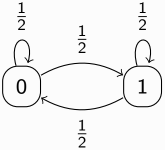


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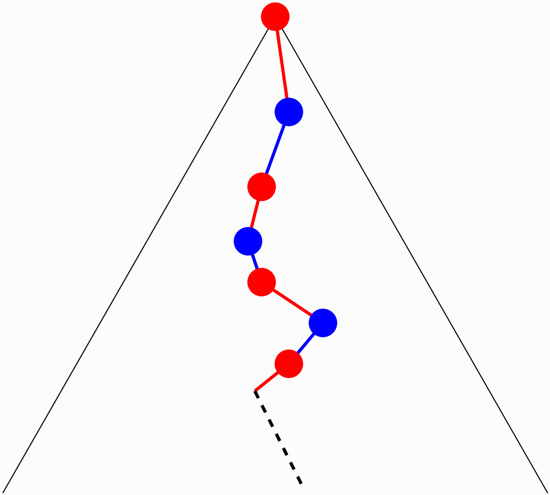
One can also prove the non existence of winning move-counting strategy

# Outline of the talk

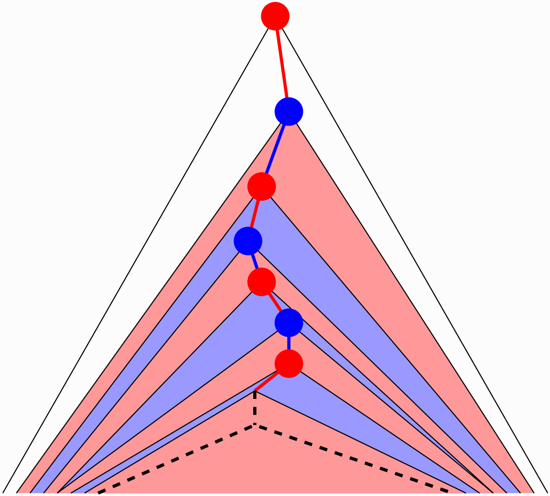
# Banach-Mazur game

A play consists in concatenating **finite paths**,



# Banach-Mazur game

A play consists in concatenating **finite paths**,
or equivalently in building a decreasing sequence of **open sets**.



# Another simple strategy

Given $\mathcal{G} = (G, v_0, W)$, a strategy for Pl. 0 can be seen as $f : \mathcal{O}^* \to \mathcal{O}$.

$$f(\underbrace{O_1 O_2 \cdots O_{2n+1}}_{\text{What is observed}}) = \underbrace{O_{2n+2}}_{\text{What is played}},$$

where $O_1 \supseteq O_2 \supseteq \cdots \supseteq O_{2n+1} \supseteq O_{2n+2}$ are open sets.

Assuming that $G$ is equipped with a probability distribution on edges.

**The notion of $\alpha$-strategy**

Given $0 < \alpha < 1$, we say that $f$ is an *$\alpha$-strategy* if and only if

$$\mathbb{P}(O_{2n+2} | O_{2n+1}) \geqslant \alpha.$$

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# Our results

**Theorem**

Let $\mathcal{G} = (G, v_0, W)$ be a Banach-Mazur game on a finite graph and $\mathbb{P}$ a reasonable probability measure.

If Pl. 0 has a winning $\alpha$-strategy for some $\alpha > 0$, then $\mathbb{P}(W) = 1$.

**Theorem**

When $W$ is a **countable intersection of open sets**, the following assertions are equivalent :

1. $P(W) = 1$,
2. Pl. 0 has a winning $\alpha$-strategy for some $\alpha > 0$,
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# Summary


Winning *positional* strategy
Winning *finite memory* strategy
Winning *bounded* strategy
Winning *move-counting* strategy
Winning $\alpha$-strategy
Countable intersection
of open sets
Probability 1
Winning *length-counting* strategy
Winning strategy

Thank you ! ! !